1 Higher Homotopy Groups

Definition I. Let $n \geq 1$ and let $I^n = [0,1]^n$ be the n-dimensional cube. Its boundary is the subspace

$$\partial I^n = \{(x_1, \ldots, x_n) \mid x_i \text{ is 0 or 1 for some } i\}.$$

Let (X, x_0) be a based space. Then we define the *nth homotopy group of* (X, x_0), denoted $\pi_n(X, x_0)$, to be a group of equivalence classes of maps $I^n \to X$ that map ∂I^n to x_0. Two maps are equivalent if they are homotopic via a homotopy that maps ∂I^n to x_0 at all times.The group operation is induced by concatenation of maps,

$$(f \bullet g)(s_1, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n), & s_1 \in \left[0, \frac{1}{2}\right] \\ g(2s_1 - 1, s_2, \ldots, s_n), & s_1 \in \left[\frac{1}{2}, 1\right] \end{cases}$$

In other words, $\pi_n(X, x_0)$ is the group of homotopy classes (rel ∂I^n) of maps $I^n \to X$ mapping ∂I^n to x_0.

Exercise 1. (Bonus) Let (X, x_0) be a based space.

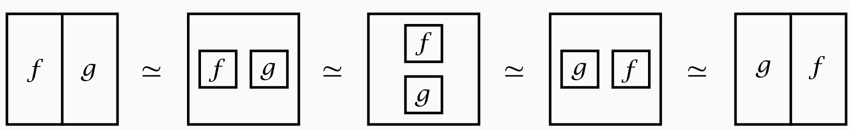
(a) Explain the sense in which every map $(I^n, \partial I^n) \to (X, x_0)$ corresponds to a based map $(S^n, s_0) \to (X, x_0)$ from the n-sphere S^n, and every element of $\pi_n(X, x_0)$ corresponds to a homotopy class of based maps $(S^n, s_0) \to (X, x_0)$ rel s_0. Interpret the group operation in this framework.

(b) Verify that the concatenation product on $\pi_n(X, x_0)$ is well-defined on equivalence classes.

(c) Construct an identity element, and inverse elements, for the product operation.

(d) Verify that the product on $\pi_n(X, x_0)$ defines a group structure.

Exercise 2. The following figure (from Hatcher) is presented as an argument that $\pi_n(X, x_0)$ is an *abelian* group for all $n \geq 2$. Explain this figure and this argument. Why doesn't the argument apply to $\pi_1(X, x_0)$?



For this reason, many sources use additive notation $+$ for the group operation on $\pi_n(X, x_0)$ for $n \geq 2$.

Definition II. By convention, we let $\pi_0(X, x_0)$ denote the set of path-components of X.

In general this set does not have a suitable group structure.

Exercise 3. (Bonus) By convention, I^0 is a point. Show that we can identify $\pi_0(X, x_0)$ with the set of equivalence classes of maps $(I^0, \varnothing) \to (X, x_0)$; alternatively, with the set of homotopy classes of based maps $(S^0, s_0) \to (X, x_0)$.

Proposition III. *For all $n \geq 1$ the assignment $(X, x_0) \mapsto \pi_n(X, x_0)$ defines a functor from based spaces to groups.*

Proposition IV. *Homotopy groups are homotopy invariants.*

Proposition V. *Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering space map. The map $p_* : \pi_n(\tilde{X}, \tilde{x_0}) \to \pi_n(X, x_0)$ is an isomorphism for all $n \geq 2$.*

In particular, spaces with universal covers have isomorphic higher homotopy groups to their universal covers'.

Exercise 4. (Bonus) In this problem we prove Propositions III and IV. Fix $n \geq 1$.

(a) Verify that π_n is functorial with respect to based maps of based spaces $f : (X, x_0) \to (Y, y_0)$. Define the induced map $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$, verify that it is a homomorphism of groups, and verify that the assignment $f \mapsto f_*$ respects identity and composition of maps.

(b) Prove that if maps $f, g : (X, x_0) \to (Y, y_0)$ are homotopic, then $f_* = g_*$.

(c) Deduce that a homotopy equivalence $(X, x_0) \to (Y, y_0)$ induces an isomorphism $\pi_n(X, x_0) \cong \pi_n(Y, y_0)$.

Exercise 5. Prove Proposition V. What does this tell you about the homotopy groups of graphs, n-tori, . . . ?

Exercise 6. Let $\prod_\alpha X_\alpha$ be a product of path-connected spaces X_α. Show that $\pi_n\left(\prod_\alpha X_\alpha\right) \cong \prod_\alpha \pi_n(X_\alpha)$.

Proposition VI. *Let X be a path-connected space. Let $n \geq 1$. Then there are (non-canonical) isomorphisms of groups $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ for any $x_0, x_1 \in X$.*

We write $\pi_n(X)$ for this abstract group.

Exercise 7. (Bonus) Let $n \geq 2$. Let (X, x_0) be a based space. Let x_1 be another point in the path component of x_0, and $\gamma : I \to X$ a path from x_1 to x_0 with inverse $\overline{\gamma}(t) := \gamma(1 - t)$.

(a) Describe how we can use γ to define a change-of-basepoint map $\beta_\gamma : \pi_n(X, x_1) \to \pi_1(X, x_0)$. Verify that your construction is well-defined on equivalence classes.

(b) Show that, if γ is a constant path, then β_γ is the identity map.

(c) Verify that β_γ only depends on the homotopy class (rel the endpoints of I) of γ.

(d) Verify that β_γ is a group homomorphism.

(e) Verify $\beta_\gamma \circ \beta_{\gamma'} = \beta_{\gamma \bullet \gamma'}$. In particular, β_γ is an isomorphism with inverse $\beta_{\overline{\gamma}}$, which proves Proposition VI.

(f) Consider the special case that $x_0 = x_1$. Show that the assignment $\gamma \mapsto \beta_\gamma$ gives a well-defined homomorphism of groups $\pi_1(X, x_0) \to \text{Aut}(\pi_n(X, x_0))$. This defines the *action of π_1 on π_n*.

Definition VII.

- All spaces are (-2)-*connected*.
- A space is (-1)-*connected* if it is nonempty.
- A space is 0-*connected* if it is path-connected.
- A space is 1-*connected* if it is simply connected.
- In general, for $n \geq 0$, a space X is n-*connected* if it is nonempty and $\pi_i(X) \cong 0$ for all $0 \leq i \leq n$.
- X is *weakly contractible* if $\pi_i(X) \cong 0$ for all $i \geq 0$.

A pair (X, A) is n-*connected* if the inclusion $A \hookrightarrow X$ induces isomorphisms on π_k for $k < n$ and surjection on π_n.

Proposition VIII shows a space $X \neq \varnothing$ is n-connected iff every map $S^i \to X$ is nullhomotopic for all $0 \leq i \leq n$.

Proposition VIII. *Let X be a space. The following are equivalent.*

(i) Every map $S^n \to X$ is homotopic to a constant map.

(ii) Every map $S^n \to X$ extends to a map $D^{n+1} \to X$.

(iii) $\pi_n(X, x_0) = 0$ for all x_0 in X.

Exercise 8. Prove Proposition VIII. Take care with basepoints and relative homotopies!