1 Whitehead's Theorem

Theorem I (Whitehead's Theorem)**.** *Let X, Y be connected CW complexes.*

(i) If $\iota : X \to Y$ is the inclusion of subcomplex with the property that $\iota_* : \pi_n(X) \xrightarrow{\cong} \pi_n(Y)$ is an isomorphism for all $n \geq 0$, then Y deformation retracts onto the subcomplex X.

(ii) If $f : X \to Y$ is any map that induces isomorphisms $f_* : \pi_n(X) \xrightarrow{\cong} \pi_n(Y)$ for all $n \geq 0$, then f is a homotopy equivalence.

Exercise 1. (Bonus) Show by example that a map $f : X \to Y$ of CW complexes that induces isomorphisms $f_* : H_n(X) \to H_n(Y)$ on homology groups for all $n \geq 0$ need not be a homotopy equivalence.

Exercise 2. In this problem we will prove part (i) of Whitehead's Theorem. Suppose $\iota : X \to Y$ is an inclusion of connected CW complexes. We wish to construct a deformation retraction $F : Y \times I \to Y$ of Y onto X, and we will proceed by induction on the skeleta of Y. Specifically, we wish to construct homotopies

$$F^{(n)} : Y \times I \longrightarrow Y$$

satisfying the following properties

- When $n = 0$, the map $F^{(0)}\big|_{t=0}$ is the identity map $\mathrm{id}_Y : Y \to Y$.
- When $n \geq 1$, $F^{(n)}\big|_{t=0} = F^{(n-1)}\big|_{t=1}$
- The image of $(X \cup Y^{(n)})$ under $F^{(n)}\big|_{t=1}$ is contained in X.
- $F^{(n)}$ is stationary on $(X \cup Y^{(n-1)})$, that is, its restriction is independent of t.

(a) Explain why it is possible to define a deformation retraction from $Y^{(0)} \cup X$ to X. Explain how to use the homotopy extension property to construct $F^{(0)}$.

(b) Suppose $\iota_* : \pi_d(X) \to \pi_d(Y)$ is surjective and $\iota_* : \pi_{d-1}(X) \to \pi_{d-1}(Y)$ is injective. Prove that any map $(D^d, \partial D^d) \to (Y, X)$ can be homotoped rel ∂D^d to a map with image contained in X.

(c) Now suppose $\iota_* : \pi_n(X) \xrightarrow{\cong} \pi_n(Y)$ is an isomorphism for all n, and suppose we have constructed our homotopy $F^{(k)}$. Explain how to construct the homotopy $F^{(k+1)}$ on $Y^{(k+1)} \cup X$.

(d) Explain how to extend the homotopy $F^{(k+1)}$ over Y.

(e) Explain the sense in which we can concatenate the homotopies $F^{(n)}$ to obtain a homotopy $F : Y \times I \to Y$, by running the kth step during time interval $[1 - \frac{1}{2^k}, 1 - \frac{1}{2^{k+1}}]$. Verify that this construction is continuous.

(f) Verify that the homotopy F is a deformation retraction from Y to X. This concludes the proof.

Exercise 3. (Bonus) Prove part (ii) of Whitehead's theorem. *Hint:* Reduce to part (i) by considering the inclusion of X into the mapping cylinder M_f of $f : X \to Y$. See Hatcher Theorem 4.5.

Recall that a space is *weakly contractible* if its homotopy groups vanish in all degrees.

Corollary II. *A weakly contractible CW complex is contractible.*

Exercise 4. (a) Prove Corollary II.

(b) **(Bonus)** Show that the assumption that the space is a CW complex is necessary. *Hint:* Consider the long line.

Exercise 5. Let X be a connected CW complex. Show that $\pi_n(X) \cong 0$ for all $n \geq 2$ if and only if the universal cover for X is contractible.

Exercise 6. (a) **(Bonus)** Show that $\mathbb{R}\mathrm{P}^2$ and $S^2 \times \mathbb{R}\mathrm{P}^\infty$ have isomorphic homotopy groups.
Hint: Consider their universal covers.

(b) **(Bonus)** Show that $\mathbb{R}\mathrm{P}^2$ and $S^2 \times \mathbb{R}\mathrm{P}^\infty$ are not homotopy equivalent. *Hint:* Consider their homology groups.

(c) Explain why this does not contradict Whitehead's theorem.