1 Review: The Cellular Approximation Theorem

Definition I. A continuous map $f : X \to Y$ of CW complexes X, Y is called *cellular* if $f\left(X^{(n)}\right) \subseteq Y^{(n)}$ for all n.

Despite the name, a cellular map does not need to map individual cells to cells. For this reason some authors instead use the term *skeletal* map.

Theorem II (Cellular Approximation)**.** *Every continuous map $f : X \to Y$ of CW complexes is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subseteq X$, the homotopy may be taken to be stationary on A.*

Corollary III. *Let X be a CW complex. Then the pair $\left(X, X^{(n)}\right)$ is n-connected, that is, the map $\pi_k\left(X^{(n)}\right) \to \pi_k(X)$ induced by the inclusion of the n-skeleton $X^{(n)} \hookrightarrow X$ is an isomorphism for $k < n$ and surjective for $k = n$. More generally, a pair (X, A) is n-connected if the complement $X \setminus A$ consists of cells of dimension strictly greater than n.*

In particular,

- To prove X is n-connected it suffices to show $X^{(n)}$ is.
- X is n-connected if and only if $X^{(n+1)}$ is.
- If X is n-connected, then $X^{(k)}$ is $(k-1)$-connected for all $k = 0, 1, 2, \ldots, n$.
- $\pi_k\left(X^{(k+1)}\right) \cong \pi_k\left(X^{(k+2)}\right) \cong \pi_k\left(X^{(k+3)}\right) \cong \ldots \cong \pi_k(X)$.

Corollary IV. *The n-sphere S^n is $(n-1)$-connected:*

$$\pi_k(S^n) \cong 0 \qquad \text{for all } k < n.$$

Exercise 1. Assuming Theorem II, prove Corollaries III and IV.

Exercise 2. Some sources define a finite CW complex as any topological space X such that there exists a finite nested sequence

$$\varnothing \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = X$$

where, for each $i = 0, 1, \ldots, n$, the space X_i is the result of attaching a disk to X_{i-1} along its boundary via any continuous attaching map. This definition has no requirement that the attaching map takes ∂D^d to cells of dimension less than d.

(a) Explain why a space X constructed in this way is homotopy equivalent to CW complex in the sense of the standard definition. In particular, up to homotopy equivalence we can assume that the cells are added so that their dimensions are nondecreasing.

(b) Suppose that a space X is built inductively in this way from one 0-cell and a nonzero finite collection of d-cells, with no conditions (other than continuity) on the attaching maps. Verify that X is homotopy equivalent to a wedge of d-spheres.

Exercise 3. (Bonus) In this exercise we will give a partial proof of the cellular approximation theorem, omitting some significant technical detail. Let X, Y be CW complexes, $f : X \to Y$ a continuous map, and $A \subseteq X$ a subcomplex such that $f\big|_A$ is cellular.

(a) Assume by induction that f has been homotoped rel A to be cellular on $X^{(n-1)}$. Let e^n be an n-cell of X. Explain why its image $f(e^n)$ meets only finitely many cells of Y. Thus there is a cell e^d of Y that intersects $f(e^n)$ that is maximal dimensional among all such cells.

(b) If $d \leq n$, we are done. Assume $d > n$. It is possible that $f\big|_{e^n}$ is a space-filling curve. It is a nontrivial result (we will not prove) that it is possible to homotope $f\big|_{X^{(n-1)} \cup e^n}$ rel $X^{(n-1)}$ so that its image misses a point p in e^d. Explain why we can homotope $f\big|_{X^{(n-1)} \cup e^n}$ rel $X^{(n-1)}$ so that its image does not intersect e^d.

(c) Explain how to construct a homotopy of $f\big|_{X^{(n)}}$ rel $\left(X^{(n-1)} \cup A^{(n)}\right)$ to a cellular map.

(d) Explain how to construct a homotopy of f rel A so that $f\big|_{X^{(n)}}$ is cellular.

(e) Verify the base case.

(f) Explain how (if X is infinite dimensional) to implement this construction to homotope f to a cellular map.

See Hatcher Theorem 4.8 and Lemma 4.10 for the remaining details.