1 The Hurewicz Theorem

Let X be a topological space. Observe that there is (at least up to sign) a natural map $h_n : \pi_n(X, x_0) \to H_n(X)$ as follows. Given an element $[\gamma] \in \pi_n(X, x_0)$ with representative $\gamma : (S^n, s_0) \to (X, x_0)$, and given a choice of generator u for $H_n(S^n) \cong \mathbb{Z}$, we map $[\gamma]$ to the image of u under the induced map $\gamma_* : H_n(S^n) \to H_n(X)$,

$$\begin{aligned} h_n : \pi_n(X, x_0) &\longrightarrow H_n(X) \\ [\gamma] &\longmapsto \gamma_*(u) \end{aligned}$$

This map is called the *Hurewicz homomorphism*.

Theorem I (Hurewicz Theorem)**.** *Let X be a space, $A \subseteq X$ and $x_0 \in X$.*

(i) If X is path-connected, the Hurewicz map $\pi_1(X, x_0) \to H_1(X)$ is surjective and identifies $H_1(X)$ with the abelianization of $\pi_1(X, x_0)$.

(ii) Let $n \geq 2$. If X is $(n-1)$-connected, then $H_i(X) \cong 0$ for $i < n$ and when $i = n$ the Hurewicz map is an isomorphism $\pi_n(X) \cong H_n(X)$.

(iii) Let $n \geq 2$. Assume A is 1-connected (nonempty and simply-connected). Suppose the pair (X, A) is $(n-1)$-connected, that is, the inclusion $A \hookrightarrow X$ induces isomorphisms on π_i for $i < n$ and a surjection for $i = n$. Then $H_i(X, A) = 0$ for $i < n$.

(iv) Let $n \geq 2$. Suppose X is a CW complex and A a subcomplex. Assume A is 1-connected. If the pair (X, A) is $(n-1)$-connected, then $H_i(X/A) = 0$ for $i < n$ and for $i = n$ there is an isomorphism $\pi_n(X/A) \cong H_n(X/A)$.

The Hurewicz theorem says, in particular, that if X is simply connected, then the first nonzero homology and homotopy groups appear in the same degree, and are isomorphic.

Given a map $f : X \to Y$ between simply connected CW complexes (note the assumption on π_1), we can apply the Hurewicz and Whitehead theorems to the mapping cylinder of f to deduce the following corollary.

Corollary II (Whitehead's theorem for homology)**.** *Let X and Y be simply connected CW complexes. If a map $f : X \to Y$ induces isomorphisms on homology $f_* : H_n(X) \xrightarrow{\cong} H_n(Y)$ for all n, then f is a homotopy equivalence.*

Corollary III. *Let X be a simply connected CW complex. If the reduced homology of X vanishes in all degrees, then X is contractible.*

Corollary IV. *Let X be a simply-connected CW complex and $n \geq 2$. Suppose the reduced homology of X is nonzero in exactly one degree (degree n), and is free abelian,*

$$\widetilde{H}_k(X) \cong \begin{cases} 0, & k \neq n \\ \bigoplus_I \mathbb{Z}, & k = n. \end{cases}$$

Then X is homotopy equivalent to a wedge of n-spheres,

$$X \simeq \bigvee_I S^n.$$

Corollary V. *Let $n \geq 1$. Any $(n-1)$-connected CW complex of dimension n is either contractible or homotopy equivalent to a wedge of n-spheres.*

Exercise 1. (Bonus) Verify that the Hurewicz map is well-defined, and a homomorphism.
Hint: First describe the group operation in $\pi_n(X, x_0)$ as a composite of maps

$$S^n \longrightarrow S^n \vee S^n \longrightarrow X \vee X \longrightarrow X.$$

Exercise 2. (Bonus) Can you formulate a version of Theorem I part (ii) when $n = 0, -1$?

Exercise 3. (Bonus) Let X be a simply-connected space. Show by example that we cannot expect any nice relationships between the homotopy and homology groups of X beyond the degree of the first nonzero groups.
Hint: Consider $X = S^n$.

Exercise 4. Assuming Corollary II, prove Corollaries III, IV, and V.

Exercise 5.

(a) Suppose that a CW complex X is $(d-1)$-connected. Prove that, for $0 \leq k \leq d$ its k-skeleton is either contractible or homotopy equivalent to a wedge of k-spheres.

(b) **(Bonus)** As an application, compute the homotopy type of the k-skeleton of a standard n-simplex.

Exercise 6. (Bonus) In this exercise, we will prove part (ii) of the Hurewicz Theorem. Suppose X is an $(n-1)$-connected CW complex for some fixed $n \geq 2$.

(a) Using Whitehead's theorem, construct a homotopy equivalence $\check{X} \to X$ between X and a CW complex $\check{X}$ whose $(n-1)$-skeleton is a point. Verify that this can be done in such a way that the attaching maps ϕ_β^{n+1} of the $(n+1)$-cells are based maps with respect to the unique 0-cell.

(b) Deduce $\widetilde{H}_i(X) \cong \widetilde{H}_i(\check{X}) \cong 0$ for $i < n$.

(c) Explain why we can replace $\check{X}$ by its $(n+1)$-skeleton $\check{X}^{(n+1)}$ without changing its degree-n homotopy or homology groups.

(d) Verify that $\check{X}^{(n+1)}$ has the form $\bigvee_\alpha S^n_\alpha \cup_\beta D^{n+1}_\beta$.

(e) Prove that $\pi_n(X) \cong \pi_n(\check{X}^{(n+1)})$ is isomorphic to the cokernel of the cellular boundary map

$$d : H_{n+1}\left(\check{X}^{(n+1)}, \check{X}^{(n)}\right) \to H_n\left(\check{X}^{(n)}, \check{X}^{(n-1)}\right).$$

(f) Explain why $H_n(X) \cong H_n(\check{X}^{(n+1)})$ is equal to the cokernel of the map d. This concludes the proof in the case that X is a CW complex.

(g) Use the CW approximation theorem to conclude the result for a general topological space X.

See Hatcher Theorem 4.32 for details.