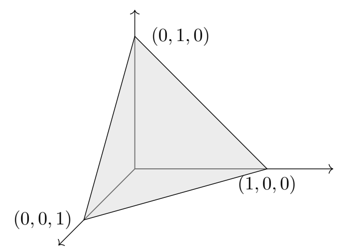
1 Δ-complexes and simplicial complexes

Definition I. For $n \geq 1$ we define *the standard n-simplex* Δ^n to be the convex hull of the $(n+1)$ standard basis vectors in $\mathbb{R}^{n+1}$. Concretely,

$$\Delta^n = \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} t_i \geq 0 \text{ for all } i, \\ t_0 + t_1 + \cdots + t_n = 1 \end{array} \right\}.$$

We define Δ^0 to be a point.
More generally, an *n-simplex* is a closed n-disk identified homeomorphically with the standard n-simplex.



This parameterization of an n-disk allows us to study linear maps between simplices. For an alternative coordinate system on an n-simplex see Exercise 2.

Definition II. Given a subset $A \subseteq \{0, 1, 2, \dots, n\}$ of cardinality $(m+1)$ for $m \geq 0$, the subset

$$\{(t_0, t_1, \dots, t_n) \in \Delta^n \mid t_i = 0 \text{ for all } i \notin A\}$$

is called a *face* or *m-face* of Δ^n. The 0-faces, called the *vertices* of Δ^n, are the standard basis elements $(1, 0, 0, \dots, 0)$, $(0, 1, 0, 0, \dots, 0), \dots$. The 1-faces are called *edges*. The *boundary* of the simplex is the union of its proper faces and is denoted $\partial\Delta^n$.

Proposition III. *For fixed m, n, let f be a map from the vertex set $v_0, v_1, \dots, v_m$ of Δ^m to the vertex set of Δ^n. Then f extends uniquely to a linear map*

$$f : \Delta^m \longrightarrow \Delta^n$$
$$t_0 v_0 + t_1 v_1 + \cdots + t_m v_m \longmapsto t_0 f(v_0) + t_1 f(v_1) + \cdots + t_m f(v_m)$$

The m-faces of a n-simplex are linearly isomorphic to m-simplices by a map of this form.

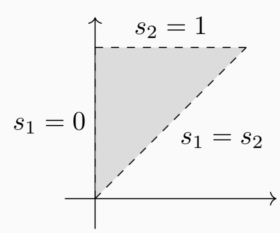
Exercise 1. (Bonus) Verify that Δ^1 is a closed line segment, Δ^2 is a triangle, and Δ^3 is a tetrahedron. Can you describe Δ^4?

Exercise 2. (Bonus)

Some sources define the standard n-simplex as the subset of $\mathbb{R}^n$ (not $\mathbb{R}^{n+1}$) given by

$$\Delta^n_* = \{(s_1, s_2, \dots s_n) \mid 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq 1\}$$

Unlike Definition I, this construction is not symmetric under permutations of the coordinates.



(a) Verify that the map $s_i = t_0 + t_1 + \cdots + t_i$ defines a linear isomorphism between the two definitions of an n-simplex.

(b) Describe the m-dimensional faces of Δ^n_*, and verify that they are linearly isomorphic to m-simplices.

(c) Describe how a map between vertex sets of two simplices extends to a linear map of simplices with this parameterization.

Exercise 3. (Bonus) Verify Proposition III.

Exercise 4. (Bonus) Check that the Cartesian product of simplices does not in general inherit a natural simplex structure. Can you devise a convention for subdividing the product $\Delta^n \times \Delta^m$ into a union of simplices?

Definition IV. A Δ-*complex* is a CW complex X for which the n-disks are parameterized as n-simplices Δ^n_α, and the attaching maps $\phi^n_\alpha : \partial\Delta^n_\alpha \to X^{(n-1)}$ restrict to a linear isomorphism from each $(n-1)$-face of Δ^n_α to an $(n-1)$-simplex Δ^{n-1}_β (in the sense of Proposition III) composed with a characteristic map $\Delta^{n-1}_\beta \to X^{(n-1)}$.

Some authors require that Δ-complexes have the additional structure of an ordering on the vertices of each simplex that is compatible with restriction to faces. These orderings are necessary for computing simplicial homology. We can always construct compatible orderings on the simplices of a Δ-complex by first choosing a total ordering on the 0-simplices.

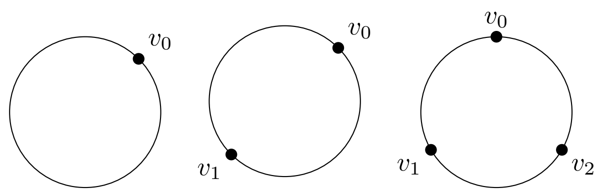
Definition V. A *simplicial map* of Δ-complexes $f : X \to Y$ is a map that restricts on each simplex of X to a linear map of simplices in the sense of Proposition III.

By Worksheet #2 Corollary V, simplicial maps are continuous.

Definition VI. A *simplicial complex* is a Δ-complex subject to the additional conditions,

- Each n-simplex has $(n+1)$ distinct vertices.
- Any $(n+1)$ vertices span at most one n-simplex.

Example VII. The following figure shows three Δ-complex structures on the circle S^1. Only the third is a simplicial complex structure.



An (open) cell in a Δ-complex is sometimes called an *open simplex*. A closed n-cell in a simplicial complex is an embedded n-simplex, though in a Δ-complex a closed n-cell may be a quotient of a n-simplex with one or more of its faces glued together via linear isomorphisms.

Exercise 5. (Bonus) Show that an n-simplex has the structure of a simplicial complex, in which the m-skeleton is the union of the m-faces.

Exercise 6. (Bonus) Show that for a simplicial complex—but not necessarily for a Δ-complex—each n-simplex Δ^n_α embeds in the complex.

Exercise 7. (Bonus) Find a simplicial structure on the torus. Any simplicial structure includes at least fourteen 2-simplices.

Exercise 8. (Bonus) Some sources define a simplicial complex as a union of simplices $\mathcal{K}$ subject to the conditions:

- Every face of a simplex in $\mathcal{K}$ is a simplex in $\mathcal{K}$.
- The intersection of two simplices σ_1, σ_2 in $\mathcal{K}$ must either be empty or equal a common face of both σ_1 and σ_2.

Reconcile this definition with our definition.