1 Abstract simplicial complexes

In a simplicial complex, every simplex is uniquely determined by its vertex set. Hence, the data of the simplicial complex is completely specified by the vertex set, and the collection of its subsets that span simplices. This motivates the following definition.

Definition I. An *abstract simplicial complex* X consists of the following data.

- a set $V(X)$ called the *vertices*,
- a set $S(X)$ consisting of finite nonempty subsets of $V(X)$, called the *simplices*,

subject to the following conditions:

- If $\sigma \subseteq V(X)$ is a simplex and $\tau \subseteq \sigma$, then τ is a simplex. The subset τ is called a *face* of σ.
- For each $v \in V(X)$, the singleton $\{v\}$ is a simplex.

A subset $\sigma \in S(X)$ of cardinality $(p+1)$ is called a p-simplex. The set of p-simplices is denoted by X_p. The *dimension* of X is the maximal value of p such that $S(X)$ contains a p-simplex; it may be infinity.

Remark II. Some authors, including Kozlov, define $S(X)$ to include the empty set $\varnothing$.

Exercise 1. (Bonus)

(a) Show that any simplicial complex uniquely specifies an abstract simplicial complex.

(b) Conversely, explain how an abstract simplicial complex determines the n-cells and attaching map data of a simplicial complex.

Exercise 2. (Bonus) Write down the abstract simplicial complexes that correspond to ...

(a) The simplex Δ^n with its standard simplicial structure

(b) Your favourite simplicial structures on S^1 and S^2.

(c) Your favourite simplicial structures on $\mathbb{R}$ and $\mathbb{R}^2$.

Definition III. Let X be an (abstract) simplicial complex. A simplex of X is called *maximal* if it not strictly contained in any other simplex. For $d < \infty$, the complex X is called *pure of dimension* d if every simplex is contained in a simplex of dimension d, equivalently, all maximal simplices of X have the same dimension d.

Definition IV. Given an abstract simplicial complex X, the corresponding simplicial complex—the topological space with the simplicial CW structure specified by X—is called *geometric realization* of X. It is denoted $|X|$.

Definition V. A *simplicial map* $X \to Y$ of abstract simplicial complexes is a function of their vertex sets $f : V(X) \to V(Y)$ with the property that, whenever $\sigma \subseteq V(X)$ is a simplex of X, then $f(\sigma)$ is a simplex of Y.

Proposition VI. *A simplicial map of abstract simplicial complexes $X \to Y$ induces a continuous simplicial map $|X| \to |Y|$ on their geometric realizations. Conversely, every simplicial map $|X| \to |Y|$ arises in this way.*

Exercise 3. (a) Verify Proposition VI.

(b) Let X and Y be Δ-complexes, and let f be a map of their vertex sets. Formulate an analogue of the condition of Definition V. Does this condition imply that f always extends to a continuous map? Does it extend uniquely?

2 Subcomplexes of abstract simplicial complexes

Definition VII. A *subcomplex* A of an abstract simplicial complex X is a subset of $V(X)$ and a subset of $S(X)$ that itself satisfies the conditions of a simplicial complex.

Exercise 4. Suppose A is a subcomplex of the abstract simplicial complex X. Show that $|A|$ is naturally a subcomplex of the simplicial complex $|X|$, and conversely every subcomplex arises in this way.

Given an abstract simplicial complex and a collection of simplices $B \subseteq S(X)$, the *closure* of B is the smallest subcomplex containing B, that is, the collection of all nonempty subsets of elements of B. From this perspective we may think of elements of $S(X)$ as corresponding to open simplices in X, and the closure as corresponding to the topological closure.

Definition VIII. A subcomplex A of a simplicial complex X is *full* if, whenever vertices $v_0, \ldots, v_p$ are vertices of A, and $\{v_0, \ldots, v_p\}$ is a simplex of X, then $\{v_0, \ldots, v_p\}$ is a simplex of A.

2.1 Some important subcomplexes

Definition IX. Let X be an abstract simplicial complex, and σ a simplex of X. The (closed) *star* of σ is the subcomplex consisting of the closures of all simplices containing σ as a face.

$$\mathrm{Star}_X(\sigma) = \{\tau \in S(X) \mid \tau \cup \sigma \in S(X)\}.$$

The *deletion* of σ in X is the subcomplex

$$\mathrm{dl}_X(\sigma) = \{\tau \in S(X) \mid \sigma \not\subseteq \tau\}$$

The *link* of σ in X is the subcomplex

$$\mathrm{Lk}_X(\sigma) = \{\tau \in S(X) \mid \sigma \cup \tau \in S(X),\ \sigma \cap \tau = \varnothing\}$$

Exercise 5. Explain, in words and in pictures, the subcomplexes of a (geometric) simplicial complex that correspond to the star, deletion, and link of a simplex.

Exercise 6. Let σ be a simplex of X. Discuss: under what conditions are the star, deletion, and link of σ a full subcomplex of X?

Exercise 7. Prove or disprove the following equalities.

(a) $\mathrm{Lk}_{\mathrm{Lk}_X(\sigma)}(\tau) = \mathrm{Lk}_X(\sigma \cup \tau)$ (b) $\mathrm{Lk}_X(\sigma) = \bigcap_{v \in \sigma} \mathrm{Lk}_X(v)$ (c) $\mathrm{Lk}_X(\sigma) = \mathrm{Star}_X(\sigma) \cap \mathrm{dl}_X(\sigma)$

Definition X. **(Bonus)** Let X, Y be abstract simplicial complexes. Let v_X and v_Y be vertices of X and Y, respectively. Describe the abstract simplicial complex that corresponds to the wedge sum of simplicial complexes $|X| \vee |Y|$ by identifying the vertices v_X and v_Y.