Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Give an intuitive geometric explanation of each of the 3 properties that define a metric.

2. Let $X = \{a, b, c\}$. Which of the following functions define a metric on X?

   (a)
   $$\begin{aligned} d(a,a) = d(b,b) = d(c,c) &= 0 \\ d(a,b) = d(b,a) &= 1 \\ d(a,c) = d(c,a) &= 2 \\ d(b,c) = d(c,b) &= 3 \end{aligned}$$

   (b)
   $$\begin{aligned} d(a,a) = d(b,b) = d(c,c) &= 0 \\ d(a,b) = d(b,a) &= 1 \\ d(a,c) = d(c,a) &= 2 \\ d(b,c) = d(c,b) &= 4 \end{aligned}$$

3. Which of the following functions $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the triangle inequality? Which are metrics?

   (a) $d(x,y) = \dfrac{|y-x|}{2}$

   (b) $d(x,y) = |x-y| + 1$

   (c) $d(x,y) = \min(|y-x|, \pi)$

   (d) $d(x,y) = 1$

   (e) $d(x,y) = |\log(y/x)|$ (for $x, y > 0$)

4. See Assignment Problem 1 for the definition and notation used for the image and preimage of sets under a function f.
   (a) Let X and Y be sets, and $f : X \to Y$ any function. Show that $f^{-1}(Y) = X$, and $f^{-1}(\varnothing) = \varnothing$.
   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2$. Compute $f^{-1}(\{0\})$, $f^{-1}(\{4\})$, $f^{-1}(\{-1\})$, $f((0,1))$, and $f^{-1}((0,1))$.

5. Let $f : X \to Y$. See Assignment Problem 1 for the definition of image and preimage.
   (a) Let $A \subseteq X$. Show that $y \in f(A)$ if and only if there is some $a \in A$ such that $f(a) = y$.
   (b) Let $B \subseteq Y$. Show that $x \in f^{-1}(B)$ if and only if $f(x) \in B$.
   (c) Suppose that $A \subseteq A' \subseteq X$. Show that $f(A) \subseteq f(A')$
   (d) Suppose that $B \subseteq B' \subseteq Y$. Show that $f^{-1}(B) \subseteq f^{-1}(B')$.

6. Consider the set $\mathbb{Z}$ with the Euclidean metric (defined by viewing $\mathbb{Z}$ as a subset of the metric space $\mathbb{R}$). What is the ball $B_3(1)$ as a subset of $\mathbb{Z}$? What is the ball $B_{\frac{1}{2}}(1)$?

7. Let (X, d) be a metric space, $r > 0$, and $x \in X$. Show that $x \in B_r(x)$. Conclude in particular that open balls are always non-empty.

8. Let (X, d) be a metric space, and suppose that $r, R \in \mathbb{R}$ satisfy $0 < r \leq R$. Show the containment of the subsets $B_r(x) \subseteq B_R(x)$ of X for any point $x \in X$.

9. Let (X, d) be a metric space, and $r > 0$. For $x, y \in X$, show that $y \in B_r(x)$ if and only if $x \in B_r(y)$.

10. Let (X, d) be a metric space. Let $x_0 \in X$ and $r > 0$. Let's consider the definititon of an open ball in X,
$$B_r(x_0) = \{x \in X \mid d(x, x_0) < r\}.$$
Note that the open ball (by definition) consists entirely of elements of X, it is always a subset of X. Let $Y \subseteq X$ be a susbset of X. We showed on our worksheet that Y inherits its own metric structure from tthe metric on X.
    (a) Suppose we are working with both metric spaces X and Y. Given a point $y_0 \in Y$, we can also view y_0 as a point in X. For $r > 0$, let's write
    $$B_r^Y(y_0) = \{y \in Y \mid d(y, y_0) < r\}$$
    for the open ball around y_0 in the metric space Y, and write
    $$B_r^X(y_0) = \{x \in X \mid d(x, y_0) < r\}$$
    for the open ball around y_0 in tthe metric space X. Explain why $B_r^X(y_0)$ and $B_r^Y(y_0)$ could be different sets.
    (b) Show that $B_r^Y(y_0) = B_r^X(y_0) \cap Y$.
    (c) Describe the ball of radius 2 centered around the point $y_0 = 0$ in the metric space Y, where Y is the subset of the real numbers (with the Euclidean metric)
    - $Y = \mathbb{R}$
    - $Y = [-3, 3]$
    - $Y = \left[0, \frac{1}{2}\right]$
    - $Y = [0, \infty)$
    - $Y = \mathbb{Q}$
    - $Y = \mathbb{Z}$

11. Let $X = \mathbb{R}$ with the usual Euclidean metric $d(x, y) = |x - y|$.
    (a) Let x and $r > 0$ be real numbers. Show that $B_r(x)$ is an open interval in $\mathbb{R}$. What are its endpoints?
    (b) Conversely, suppose that (a, b) is an open interval of $\mathbb{R}$ for some $-\infty < a < b < \infty$. Express (a, b) as an open ball. What is its center point and radius?
    (c) Show that every interval of the real line the form (a, b), $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$ is open, for any $a < b \in \mathbb{R}$.
    (d) Show that the interval $[0, 1] \subseteq \mathbb{R}$ is closed.

12. Let (X, d) be a metric space, and let $U \subseteq X$ be a subset. Does the set U necessarily need to be either open or closed? Can it be neither? Can it be both?

13. The interval $[0, 1]$ is not open in the metric space $X = \mathbb{R}$ with the Euclidean metric, but it **is** open in the metric space $X = [0, 1]$ with the (restriction of the) Euclidean metric. Explain this counterintuitive fact.

14. Let (X, d) be a metric space.
    (a) We saw on Worksheet #2 that a union of open subsets of X is open. Prove or find a counterexample: Given subsets $A, B \in X$, if the union $A \cup B$ is open, does this imply that A and B are open?
    (b) We will see in Assignment Problem (6) that the intersection of closed subsets is closed. Prove or find a counterexample: Given subsets $A, B \in X$, if the intersection $A \cap B$ is closed, does this imply that A and B are closed?

Worksheet problems

(Hand these questions in!)

- Worksheet #1 Problem 1(b)
- Worksheet #2 Problem 2, 3.

Assignment questions

(Hand these questions in!)

1. Let $f : X \to Y$ be a function between sets X and Y. Given a set $A \subseteq X$, its *image* under f is the subset of Y, denoted $f(A)$, defined to be

$$f(A) = \{f(a) \mid a \in A\} \subseteq Y.$$

This states that $f(A)$ is the set of all points in the codomain Y to which f maps some point of A. Given a set $C \subseteq Y$, its *preimage* is the subset of X, denoted $f^{-1}(C)$, defined to be

$$f^{-1}(C) = \{x \mid f(x) \in C\} \subseteq X.$$

This is the set of all points in the domain X that f maps to an element of C. Note that this definition makes sense (and we use the notation $f^{-1}(C)$) even if the function f is not invertible, and an inverse f^{-1} does not exist as a well-defined function of Y.

Let $f : X \to Y$, and let $A \subseteq X$ and $C \subseteq Y$. For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.

(a) $A \quad \square \quad f^{-1}(f(A)))$ (b) $C \quad \square \quad f(f^{-1}(C)))$

2. For sets X and Y, let $A, B \subseteq X$ and $C, D \subseteq Y$. Consider the Cartesian product $X \times Y$. For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.

(a) $(A \times C) \cup (B \times D) \quad \square \quad (A \cup B) \times (C \cup D)$

(b) $(A \times C) \cap (B \times D) \quad \square \quad (A \cap B) \times (C \cap D)$

(c) $(X \setminus A) \times (Y \setminus C) \quad \square \quad (X \times Y) \setminus (A \times C)$

3. Let $X = \mathbb{R}^2$. Sketch the balls $B_1(0,0)$ and $B_2(0,0)$ for each of the following metrics on $\mathbb{R}^2$. No further justification needed. Denote $\overline{x} = (x_1, x_2)$ and $\overline{y} = (y_1, y_2)$.

(a) $d(\overline{x}, \overline{y}) = ||\overline{x} - \overline{y}|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

(b) $d(\overline{x}, \overline{y}) = |x_1 - y_1| + |x_2 - y_2|$

(c) $d(\overline{x}, \overline{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

(d) $d(\overline{x}, \overline{y}) = \begin{cases} 0, & \overline{x} = \overline{y} \\ 1, & \overline{x} \neq \overline{y} \end{cases}$

4. Prove the following.

   Proposition (Equivalent definition of interior point). For a subset V of a metric space (X, d), a point $x \in V$ is an interior point of V if and only if there exists an open neighbourhood U of x that is contained in V.

5. Let $X = \mathbb{R}$ with the Euclidean metric. Recall that we proved on Worksheet #2 Example 1.6 that the subset $S = [0, 1)$ is **not** open as a subset of X.

   (a) Let $Y = [0, 2] \subseteq \mathbb{R}$ be viewed as a metric space with respect to the (restriction of the) Euclidean metric. Prove that $S = [0, 1)$ **is** open as a subset of Y.

   (b) This problem serves as a warning! Conclude that, (when we're working with both a metric space and a metric subspace) it is not enough to say a subset is "open". We need to say "open" in which metric space!