Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. (a) Give an example of a metric space and a subset that is both open and closed. Give an example of a subset that is neither open nor closed.
   (b) Recite the Topologist Scout Oath:

   "On my honour, I will do my best
   to never claim to prove a set is closed by showing that it is not open,
   and to never claim to prove a set is open by showing that it is not closed."

2. Let X be a metric space. Let U, V be subsets of X such that $U \subseteq V$. Let $x \in U$.
   (a) Suppose that x is an interior point of U. Show that x is an interior point of V.
   (b) Suppose that x is an interior point of V. Must it be an interior point of U? Give a proof or a counterexample.

3. Find all accumulation points and isolated points of the following subsets of $\mathbb{R}$ (with the Euclidean metric).

   (a) $\mathbb{R}$ (b) $\varnothing$ (c) $[0,1]$ (d) $(0,1)$ (e) $\mathbb{N}$

Worksheet problems

(Hand these questions in!)

- Worksheet # 3 Problem 1

Assignment questions

(Hand these questions in!)

1. Let $f : X \to Y$ be a function of sets X and Y. Let $C, D \subseteq Y$. For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.

   (a) $f^{-1}(C \cup D) \quad \square \quad f^{-1}(C) \cup f^{-1}(D)$ (b) $f^{-1}(C \cap D) \quad \square \quad f^{-1}(C) \cap f^{-1}(D)$

   (c) For $C \subseteq D$, $\quad f^{-1}(D \setminus C) \quad \square \quad f^{-1}(D) \setminus f^{-1}(C)$

2. **(Unions and intersections of closed sets).**
   (a) Prove *DeMorgan's Laws*: Let X be a set and let $\{A_i\}_{i \in I}$ be a collection of subsets of X.

   $$(i) \quad X \setminus \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} (X \setminus A_i) \qquad (ii) \quad X \setminus \left(\bigcap_{i \in I} A_i \right) = \bigcup_{i \in I} (X \setminus A_i)$$

   Hint: Remember that a good way to prove two sets B and C are equal is to prove that $B \subseteq C$ and that $C \subseteq B$!

(b) Let (X, d) be a metric space, and let $\{C_i\}_{i \in I}$ be a collection of closed sets in X. Note that I need not be finite, or countable! Prove that $\bigcap_{i \in I} C_i$ is a closed subset of X.

(c) Now let (X, d) be a metric space, and let $\{C_i\}_{i \in I}$ be a **finite** collection ($I = \{1, 2, \ldots, n\}$) of closed sets in X. Prove that $\bigcup_{i \in I} C_i$ is a closed subset of X.

3. **(Metric spaces are T_1).** Let (X, d) be a metric space, and let $x \in X$ be any element. Prove that the singleton set $\{x\}$ is a *closed* subset of X.
   This property is called the T_1 *property* of metric spaces. Mathematicians sometimes refer to it by the slogan "points are closed".

4. Let (X, d) be a metric space and let $S \subseteq X$ be any subset. Let S' be the set of accumulation points of S. Prove that S' is a closed subset of X.

5. Let (X, d) be a metric space and let $S \subseteq X$ be any subset. Let x be an accumulation point of S, and let $B_r(x)$ be a ball centered around x of some radius $r > 0$. Show that $B_r(x)$ contains infinitely many points of S.

6. **(Open subsets of product spaces).** Recall that, for metric spaces (X, d_X) and (Y, d_Y), the Cartesian product $X \times Y$ is metric with respect to the product metric

$$d_{X\times Y} : \ (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$$
$$d_{X\times Y}\Big((x_1, y_1), (x_2, y_2)\Big) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}.$$

(a) This part is a review of some set-theoretic properties of the product $X \times Y$, just viewed as a set. Give, with justification, an example of a subset of $X \times Y$ that is **not** of the form $A \times B$ for subsets $A \subseteq X$ and $B \subseteq Y$. Subsets of the form $A \times B$ are a special class of subsets of $X \times Y$!

(b) Prove that if $U \subseteq X$ and $V \subseteq Y$ are open sets, then $U \times V$ is an open subset of $X \times Y$.

(c) Let $W \subseteq X \times Y$ be an open set, and let $(x, y) \in W$. Show that there is a neighbourhood U_x of x in X and a neighbourhood U_y of y in Y so that $U_x \times U_y \subseteq W$.

(d) Deduce that a subset W of $X \times Y$ is open in $X \times Y$ (with respect to the product metric) if and only if it is a union of subsets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

The results of parts (b), (c), and (d) will be important later in the course when we want to show that the product metric induces the "product topology" on $X \times Y$!