Name: ______________________________ Score (Out of 12 points):

1. (5 points) Let X be a set. State the definition of a *metric* on X.

Solution. A *metric* on X is a function

$$d : X \times X \longrightarrow \mathbb{R}$$

satisfying the following conditions.

(M1) **(Positivity).** $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.

(M2) **(Symmetry).** $d(x, y) = d(y, x)$ for all $x, y \in X$.

(M3) **(Triangle inequality).** $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

2. A metric space (X, d) is called *bounded* if there exists some real number $R \geq 0$ such that $d(x, y) \leq R$ for all $x, y \in X$. In other words, any pair of points in X are distance at most R apart.

   (a) (2 points) A metric space that is not bounded is called *unbounded*. Negate the definition to give a precise statement of what it means for a metric space to be unbounded.

   Solution. A metric space (X, d) is *unbounded* if, for all real numbers $R \geq 0$, there exists points $x, y \in X$ such that $d(x, y) > R$.

   (b) (2 points) Show that the real numbers $\mathbb{R}$ with the standard Euclidean metric is unbounded.

   Solution. For any real number $R > 0$, observe that the real numbers 0 and $R + 1$ have distance $d(0, R + 1) = |R + 1 - 0| = R + 1 > R$. Thus, by part (a), $\mathbb{R}$ is unbounded.

   (c) (3 points) Let (X, d) be a metric space. Suppose that there is some element x_0 in X so that $d(x_0, x) \leq 10$ for all $x \in X$. Prove that X is bounded.

   Solution. We will prove this result by showing that $d(x, y) \leq 20$ for all $x, y \in X$.
   Let $x, y \in X$ be any two elements. Then, by the triangle inequality axiom and the symmetry axiom for the metric d,

$$\begin{aligned} d(x, y) &\leq d(x, x_0) + d(x_0, y) \\ &= d(x_0, x) + d(x_0, y) \\ &\leq 10 + 10 \\ &= 20 \end{aligned}$$

   Thus X is bounded.