Name: ______________________________ Score (Out of 6 points):

1. (6 points) Let (X, d) be a metric space. Fix $x_0 \in X$ and $r > 0$ in $\mathbb{R}$. Show that the set

$$\{x \mid d(x_0, x) \leq r\}$$

is closed. (Note the **weak** inequality in the definition of the set!)

Solution. To prove that this set is closed, by definition, we must show that its complement U is open. Observe that U is precisely the set of points

$$U = \{y \mid d(x_0, y) > r\}$$

of distance strictly greater than r from x_0. To prove that U is open, we must show that every point of U is an interior point of U.

Fix $y \in U$. Let $R = d(x_0, y) - r$. Since $d(x_0, y) > r$, we know $R > 0$. We will show that y is an interior point of U by showing that the open ball $B_R(y)$ is contained in U.

Let z be a point in $B_R(y)$. Our goal is to show that z is contained in U, that is, we must show that $d(x_0, z) > r$. By the triangle inequality, we know that

$$\begin{aligned} d(x_0, y) &\leq d(x_0, z) + d(z, y) \\ &< d(x_0, z) + R && \text{(since } z \in B_R(y)) \\ &= d(x_0, z) + d(x_0, y) - r && \text{(since } R = d(x_0, y) - r) \end{aligned}$$

Rearranging, we find that

$$d(x_0, z) > r$$

as desired. This implies that $z \in U$. Because z was an arbitrary point of $B_R(y)$, we deduce that $B_R(y) \subseteq U$.

We conclude that y is an interior point of U. We have established that U is open, and therefore that its complement $\{x \mid d(x_0, x) \leq r\}$ is closed.