

1 Open and closed sets

Definition 1.1. (Open ball of radius r about x_0 .) Let (X, d) be a metric space, and $x_0 \in X$. Let $r \in \mathbb{R}$, $r > 0$. We define the *open ball of radius r about x_0* as the subset of X

$$B_r(x_0) = \{x \in X \mid d(x_0, x) < r\} \subseteq X.$$

Example 1.2. Let $X = \mathbb{R}$ with the Euclidean metric $d(x, y) = |x - y|$. What is $B_2(-6)$?

Let $X = \mathbb{R}^2$ with the Euclidean metric. Draw $B_2(1, 0)$.

Definition 1.3. (Interior points; open sets in a metric space.) Let (X, d) be a metric space, and let $U \subseteq X$ be a subset of X . A point $x \in U$ is called an *interior point of U* if there is some radius $r_x \in \mathbb{R}$, $r_x > 0$, so that $B_{r_x}(x) \subseteq U$.

The set $U \subseteq X$ is called *open* if every point $x \in U$ is an interior point of U .

A set $A \subseteq X$ fails to be open if ...

Definition 1.4. (Neighbourhood of a point x .) Let (X, d) be a metric space, and $x \in X$. Then any open set U in X containing x is called an *open neighbourhood of x* , or simply a *neighbourhood of x* .

Proposition 1.5. *Let (X, d) be a metric space, $x_0 \in X$ and $0 < r \in \mathbb{R}$. Then the ball $B_r(x_0)$ is an open subset of X .*

Proof.

Example 1.6. Show that the interval $[0, 1) \subseteq \mathbb{R}$ is not open.

Example 1.7. Let (X, d) be a metric space. Is \emptyset open?

Definition 1.8. (Closed sets in a metric space.) A subset $C \subseteq X$ is *closed* if its complement $X \setminus C$ is open.

Warning: Despite the English connotations of the words ‘closed’ and ‘open’, mathematically, ‘closed’ does **not** mean “not open”. ‘Open’ does **not** mean “not closed”!

In-class Exercises

1. Consider \mathbb{R} with the Euclidean metric. Find an example of a subset of \mathbb{R} that is ...
 - (a) open and not closed,
 - (b) closed and not open,
 - (c) both open and closed,
 - (d) neither open nor closed.
2. Let $\{U_i\}_{i \in I}$ denote a collection of open sets in a metric space (X, d) .
 - (a) Prove that the union $\bigcup_{i \in I} U_i$ is an open set. Do not assume that I is necessarily finite, or countable!

- (b) Show by example that the intersection $\bigcap_{i \in I} U_i$ may not be open. (This means, give an example of a metric space (X, d) and a collection of open sets $U_i \subseteq X$, and prove that $\bigcap_{i \in I} U_i$ is not open).
- (c) Now assume we have a **finite** collection $\{U_i\}_{i=1}^n$ of open sets in a metric space. Prove that the intersection $\bigcap_{i=1}^n U_i$ is open.

3. Prove the following result.

Theorem 1.9. *Let (X, d) be a metric space. A subset U of X is open if and only if U can be expressed as a union of open balls in X .*

4. **(Optional).** Let (X, d) be a metric space. Let Y be a metric subspace of X , that is, Y is a subset of X that we view as a metric space with the restriction of the metric d . Let $U \subseteq Y$ be an open subset of Y . Show that there exists some open subset W of X such that $U = Y \cap W$. Conversely, if W is an open subset of X , show that $W \cap Y$ is an open subset of Y .
5. **(Optional).** Let X be a non-empty set. A function $f : X \rightarrow \mathbb{R}$ is called *bounded* if there is some number $M \in \mathbb{R}$ so that $|f(x)| \leq M$ for all $x \in X$. Let $\mathcal{B}(X, \mathbb{R})$ denote the set of bounded functions from X to \mathbb{R} .

- (a) Show that the function

$$d_\infty : \mathcal{B}(X, \mathbb{R}) \times \mathcal{B}(X, \mathbb{R}) \longrightarrow \mathbb{R}$$

$$d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|$$

is well-defined, that is, the suprema always exist.

- (b) Show that the function d_∞ defines a metric on $\mathcal{B}(X, \mathbb{R})$.
- (c) Explain why the following metric on \mathbb{R}^n is a special case of this construction.

$$d_\infty : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$d(\bar{x}, \bar{y}) = \max_{1 \leq i \leq n} |x_i - y_i|$$

where $\bar{x} = (x_1, \dots, x_n)$ and $\bar{y} = (y_1, \dots, y_n)$.

6. **(Optional).**

- (a) Rigorously verify that the sets $\{1\}$, $[1, \infty)$, and $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\} \cup \{0\}$ are all closed subsets of \mathbb{R} (with the Euclidean metric).
- (b) Consider \mathbb{R} with the Euclidean metric. Is the subset $\mathbb{Q} \subseteq \mathbb{R}$ open? Is it closed?
- (c) Let $\mathcal{C}(0, 2)$ be the set of continuous functions from the closed interval $[0, 2]$ to \mathbb{R} , with the metric

$$d_\infty : \mathcal{C}(0, 2) \times \mathcal{C}(0, 2) \longrightarrow \mathbb{R}$$

$$d(f, g) = \sup_{x \in [0, 2]} |f(x) - g(x)|$$

Determine whether the subset $\{f(x) \in \mathcal{C}(0, 2) \mid f(1) = 0\}$ is closed, open, neither, or both.

7. **(Optional).** Let X be a **finite** set, and let d be any metric on X . What can you say about which subsets of X are open? Which subsets of X are closed?