1 Continuous functions of metric spaces

Notation (Images and preimages). Let $f : X \to Y$ be a function of sets.
For a subset $A \subseteq X$, the *image* of A is the set $f(A) = \{f(a) \mid a \in A\}$, a subset of Y.
For a subset $B \subseteq Y$, the *preimage* of B is the set $f^{-1}(B) = \{x \mid f(x) \in B\}$, a subset of X.
This definition makes sense (and we use the notation $f^{-1}(B)$) even when f is not invertible.

Definition 1.1. (Continuous functions $f : \mathbb{R} \to \mathbb{R}$.) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is *continuous at a point* $x \in \mathbb{R}$ if ...

The function f is called *continuous* if it is continuous at every point $x \in \mathbb{R}$.

Rephrased:

How can we generalize this definition to general metric spaces?

Definition 1.2. (Continuous functions on metric spaces.) Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a function. Then f is *continuous at a point* $x \in X$ if ...

The function f is called *continuous* if it is continuous at every point $x \in X$.

Rephrased:

In-class Exercises

1. In this question, we will prove the following result, an equivalent definition of continuity:

   Theorem (Eqvuivalent definition of continuity). Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if, given any open set $U \subseteq Y$, its preimage $f^{-1}(U) \subseteq X$ is open.

   (a) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a continuous function. Suppose that $U \subseteq Y$ is an open set. Prove that $f^{-1}(U)$ is open.

   (b) Suppose that f is a function with the property that, for every open set $U \subseteq Y$, the preimage $f^{-1}(U)$ is an open set in X. Show that f is continuous.

2. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions between metric spaces. Show that the composite
$$g \circ f : X \to Z$$
is continuous. *Hint:* With our new criterion for continuity, this argument can be quite quick!

3. **(Optional)** Let (X, d_X) and (Y, d_Y) be metric spaces. Prove that, if X is a **finite** set, then **every** function $f : X \to Y$ is continuous.

4. **(Optional)** Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called an *isometric embedding* if it is "distance-preserving" in the sense that
$$d_Y\Big(f(x_1), f(x_2)\Big) = d_X(x_1, x_2) \qquad \text{for all } x_1, x_2 \in X.$$

   (a) Give an intuitive description, with pictures, of what it means for a map to be an isometric embedding.

   (b) Show that an isometric embedding is continuous.

   (c) Show that an isometric embedding is always injective.

   (d) Show this map is an isometric embedding of $\mathbb{R}$ into $\mathbb{R}^2$ (each with the Euclidean metric):
$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto (x, 0) \end{aligned}$$

   (e) Consider the function $f : \mathbb{R} \to \mathbb{R}^2$ given by the map $f(x) = (x, mx + b)$ for $m, b \in \mathbb{R}$, so f maps the real line to the graph of the function $mx + b$. For which values of m and b is this an isometric embedding of Euclidean spaces?

   (f) For functions $f(x) = (x, mx+b)$ that are not isometric embeddings, can you find a different parameterization of this line that is an isometric embedding? In other words, can you find an isometric embedding $g : \mathbb{R} \to \mathbb{R}^2$ whose image is the set $\{(x, mx + b) \mid x \in \mathbb{R}\}$?

   (g) Show that the image of any isometric embedding from $\mathbb{R}$ into $\mathbb{R}^2$ must be a straight line.

   (h) Let $X = \{a, b, c\}$ be a 3-point set. Find examples of metrics on X so that the resulting metric space can and cannot be isometrically embedded in Euclidean space $\mathbb{R}^2$. Can you find necessary and sufficient conditions on the metric on X to guarantee the existence of an isometric embedding of X into $\mathbb{R}^2$?