## Notation

- $I = [0, 1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit $n$-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ ($n$-sphere) (we sometimes view $S^1$ as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n\geq 1} S^n$ with the weak topology
- $\Sigma_g$ closed genus-$g$ surface
- $\mathbb{R}\mathrm{P}^n$ real projective $n$-space
- $\mathbb{C}\mathrm{P}^n$ real complex $n$-space

## Practice problems

1. Let $f : X \to Y$ be a continuous map of topological spaces.
   (a) Suppose $f$ is not surjective. Explain why $f$ factors through the inclusion of a subspace of $Y$.
   (b) Suppose $f$ is not injective. Explain why $f$ factors through a quotient of $X$.
   (c) Explain the implications of these results in algebraic topology. What do we know about $f$ if it factors through a contractible space? If we have a factorization of a map $f$, what does this imply about its induced maps on fundamental group or on homology?
   (d) Use these principles to write some possible algebraic topology exam problems.

2. (a) State the universal property for the free group $F_S$ on a set $S$, and describe our construction of $F_S$.
   (b) Formulate a universal property for the free *abelian* group on a set $S$.
   (c) Show that, for any set $S$, there exists a free abelian group $A(S)$ on $S$.
   (d) The category of abelian groups is a subcategory of the category of groups. Explain how it is possible that $A(S)$ is the "free object on $S$" in the category of abelian groups, but not in the category of all groups.
   (e) Conclude that universal properties and their universal objects really depend on which category you work in.

3. (a) Let $\mathcal{D}$ be the category of integral domains and injective ring maps. Consider the map $Frac$ from $\mathcal{D}$ to the category $\underline{\mathrm{Fld}}$ fields that takes a domain to its field of fractions. Show that $Frac$ defines a functor, by defining the action of $Frac$ on morphisms, and verifying functoriality. Why did we need to restrict $\mathcal{D}$ to *injective* ring maps?
   (b) There is forgetful functor $\mathcal{F} : \underline{\mathrm{Fld}} \to \mathcal{D}$. Describe this functor and verify functoriality.
   (c) Show that the field of fractions $Frac(D)$ of an integral domain $D$ (with its inclusion $D \to Frac(D)$) satisfies the following universal property: For every field $K$ and injective ring map $f : D \to K$, the map $f$ factors through a unique map of fields $Frac(D) \to K$. Thus we can think of $Frac(D)$ as the field "freely generated" by the domain $D$.
   (d) Deduce that there is an isomorphism (which turns out to be "natural")
   $$\mathrm{Hom}_{\mathcal{D}}(D, \mathcal{F}(K)) \cong \mathrm{Hom}_{\underline{\mathrm{Fld}}}(Frac(D), K)$$
   This says that $\mathcal{F}$ and $Frac$ are *adjoint functors*.

4. (a) Let $X$ be a nonempty set with the trivial topology $\{X, \varnothing\}$. Show that $X$ is contractible.
   (b) *Sierpiński space* is the space $S = \{0, 1\}$ with the topology $\{\varnothing, \{1\}, \{0, 1\}\}$. Show that $S$ is contractible.
   (c) Is the empty set contractible?

5. Classify the capital letters of the alphabet by homotopy type.

6. Let $A \subseteq X$ be spaces. Explain the distinction betweeen a *retraction* from $X$ to $A$, and a *deformation retraction* from $X$ to $A$.

7. Let $\Delta^n = [v_0, \dots v_n]$ be a standard $n$-simplex with its standard simplicial structure. For $k \leq n$, let $\Delta^{n,k}$ denote its $k$-skeleton.
   (a) Let $X$ be the subcomplex of $\Delta^{n,k}$ of all simplices containing the vertex $v_0$; this subcomplex is sometimes called the *star* of $v_0$. Show that $X$ is contractible.
   (b) Consider a $k$-simplex $\sigma$ of $\Delta^{n,k}$ that is not in $X$, i.e., $\sigma$ is a $k$-dimensional face of the simplex $[v_1, \dots v_n]$. Verify that the boundary of $\sigma$ is contained in $X$.
   (c) Show that the quotient $\Delta^{n,k}/X$ is a wedge of $k$-dimensional spheres, one for each $k$-dimensional face of the simplex $[v_1, \dots v_n]$.
   (d) Conclude that $\Delta^{n,k}$ has the homotopy type of wedge of $\binom{n}{k+1}$-many $k$-dimensional spheres.
   (e) Compute the Euler characteristic of $\Delta^{n,k}$ in two ways, first using this homotopy equivalence, and secondly directly from cell counts. Deduce a binomial coefficient identity.
8. Suppose that spaces $X, X'$ are homotopy equivalent, and $Y, Y'$ are homotopy equivalent. Prove or disprove: it follows that $X \times Y$ is homotopy equivalent to $X' \times Y'$.
9. Let $T$ be the torus $T = S^1 \times S^1$. Suppose we have maps
$$f : S^1 \vee S^1 \longrightarrow T \qquad \text{and} \qquad g : T \longrightarrow S^1 \vee S^1$$
   Answer, with proof, the following questions. Is it possible for $f \circ g$ to be homotopic to the identity? Is it possible for $g \circ f$ to be homotopic to the identity?
10. Let $f, g : X \to Y$ be continuous maps of spaces.
   (a) Suppose $X$ is contractible. Must $f$ and $g$ be homotopic?
   (b) Suppose $Y$ is contractible. Must $f$ and $g$ be homotopic?
   (c) Suppose $X$ is a wedge of circles and $Y$ is simply connected. Must $f$ and $g$ be homotopic?
   (d) Suppose $Y$ is a wedge of circles and $X$ is simply connected. Must $f$ and $g$ be homotopic?
11. Let $X$ be a space and $x_0 \in X$. Let $f, g : X \to Y$ be continuous maps satisfying $f(x_0) = g(x_0) = y_0 \in Y$. Let $f_*, g_*$ denote the induced maps $\pi_1(X, x_0) \to \pi_1(Y, y_0)$.
   (a) Suppose $X$ and $Y$ are homotopic via a homotopy $H$ that is stationary on $x_0$. Prove that $f_* = g_*$.
   (b) Suppose $X$ and $Y$ are homotopic via a homotopy $H$ that is **not** necessarily stationary on $x_0$. What is the relationship between $f_*$ and $g_*$? *Hint*: Let $\gamma$ denote the loop $t \mapsto H_t(x_0)$.
12. Let $D^2 = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$. Let $A \subseteq D^2$ be a subset of the disk that contains the boundary circle $S^1$ but does not contain 0. Prove that $\pi_1(A)$ contains a subgroup isomorphic to $\mathbb{Z}$.
13. Let $X$ be a connected graph. Construct maps $f, g : X \to X$ so that $f \circ g = id_X$, but $f$ and $g$ do not induce isomorphisms on $\pi_1$.
14. (a) Prove that the fundamental group of a finite CW complex is finitely generated, and therefore countable (possibly finite).
   (b) Let $Y$ be a compact space, $X$ a CW complex, and $f : X \to Y$ a continuous map. Show that the image $f_*(\pi_1(X))$ must be countable.
   (c) It is a fact (you do not need to prove) that the infinite earring has an uncountable fundamental group. Prove that the infinite earring does not have the homotopy type of a CW complex.
15. Let $G$ be a group.
   (a) Explain why $G$ is finitely generated if and only if it can be realized as the fundamental group of a connected CW complex $X$ with finite 1-skeleton.
   (b) Show by example that a finitely generated group may have infinitely generated subgroups (i.e., subgroups for which no finite generating set exists).

(c) Show that an infinitely generated subgroup of finitely generated group $G$ must have infinite index in $G$.

16. Recall that the suspension of a space $X$ is the quotient of $X \times [0,1]$ obtained by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another point. Show that, if $X$ is path-connected, then its suspension is simply connected.

17. Recall that we defined a tree to be a contractible graph. On Homework #7, we assumed a result from graph theory: a tree with $n$ vertices has $(n-1)$ edges. Use an Euler characteristic argument to prove this result.

18. Let $F_n$ be the free group on $n < \infty$ letters . Re-prove the formula giving the relationship between the index and free rank of a finite-index subgroup $G \subseteq F_n$ using Euler characteristic.

19. Prove that the free group $F_2$ on 2 generators contains a copy of the free group $F_n$ on $n$ generators for every $n \geq 2$.

20. Let $F_n$ be the free group of rank $n$. Show that $F_n$ has a finite number of index-$d$ subgroups for any $d < \infty$.

21. Let $X$ be a connected CW complex. Use homotopy extension property to explain why any continuous map $f : X \to X$ is homotopic to a map with a fixed point.

22. (a) Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $f \circ g \simeq id_Y$ and $h \circ f \simeq id_X$.

    (b) Let $X$ and $\tilde{X}$ be path-connected and locally path-connected. Let $p : \widetilde{X} \to X$ be a regular covering map, and let $\widetilde{f}$ be a map making the following diagram commute.

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "p"] & \widetilde{X} \arrow[d, "p"] \\ X \arrow[r, "id_X"] & X \end{tikzcd}$$

    Prove that $\widetilde{f}$ must be a deck transformation, that is, verify that $\widetilde{f}$ is a homeomorphism.

    (c) Let $X$ and $Y$ be path-connected, locally path-connected, semi-locally simply connected spaces, and assume they are homotopy-equivalent. Prove that their universal covers are homotopy equivalent.

23. Give an example (with proof) of a space $X$ and a subspace $A$ such that $H_n(X, A)$ is not isomorphic to $H_n(X/A)$.

24. Let $X$ be a space and $x \in X$. Is $(X, X \setminus \{x\})$ ever a good pair?

25. Let $X$ be a topological space with a finite number of path components $X_1, \ldots, X_N$. Use Mayer–Vietoris and induction to give a new calculation of the homology of $X$ in terms of the homology of the spaces $X_i$.

26. Let $\widetilde{H}_n(X)$ denote the reduced homology of a space $X$ in degree $n$. Verify that

$$\begin{aligned} \widetilde{H}_n : \underline{\text{Top}} &\longrightarrow \underline{\text{Ab}} \\ X &\longmapsto \widetilde{H}_n(X) \\ [f : X \to Y] &\longmapsto [f_* : \widetilde{H}_n(X) \to \widetilde{H}_n(Y)] \end{aligned}$$

    is a covariant functor.

27. Consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$. What can you say about the relative homology groups $H_n(\mathbb{R}, \mathbb{Q})$?

28. Define a chain homotopy, and prove that chain homotopic maps induce the same map on homology.

29. Let $M$ be a Mobius band and let $S$ be its boundary circle. Compute the homology of the quotient $M/S$ using the long exact sequence of a pair. Verify your solution by a direct analysis of the homotopy type of the topological space $M/S$.

30. Construct a connected CW complex $X$ such that $H_1(X) = 0$ but $\pi_1(X) \neq 0$.

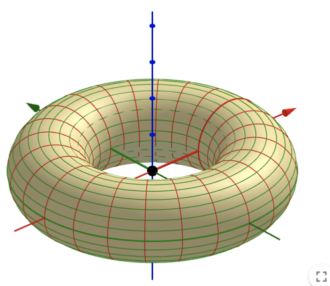
31. For $n \geq 1$, consider the map $D^n \to S^n$ defined by collapsing $\partial D^n$. Does this map admit a section?

32. Let $S^n$ be the unit sphere in $\mathbb{R}^{n+1}$, and let

$$S^{n-1} = \{(x_1, x_2, \ldots, x_n, x_{n+1}) \in S^n \mid x_{n+1} = 0\}$$

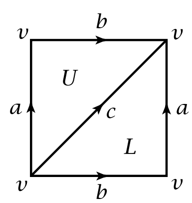
be its equator. Prove or disprove: $S^n$ retracts onto its equator.

33. Let $T$ be a smoothly embedded torus in $\mathbb{R}^3$, as shown below. Compute the homology of the quotient space $\mathbb{R}^3/T$.



34. Let $M$ be a Mobius band and let $S$ be its boundary circle. Compute the homology of the quotient $M/S$ using the long exact sequence of a pair. Verify your solution by a direct analysis of the homotopy type of the topological space $M/S$.

35. Let $T$ be the torus with the following $\Delta$-complex structure, and consider the subcomplex corresponding to the loop $a$.




Compute the relative homology groups $H_*(T, a)$ ...

(a) ... using the long exact sequence of a pair.

(b) ... by computing the quotient of simplicial chain complexes $C_*(T)/C_*(a)$ and taking homology.

36. Let $X$ be a CW complex, and let $X^k$ denote its $k$-skeleton.

(a) Show that the quotient $X^k/X^{k-1}$ is homotopy equivalent to a wedge of $k$-dimensional spheres, one for each $k$-cell of $X$.

(b) What is $H_*(X^k, X^{k-1})$?

(c) Verify your answer in the case that $X$ is a $\Delta$-complex, by describing the quotient of the simplicial chain complexes $C_*(X^k)/C_*(X^{k-1})$ and computing its homology.

37. Let $G$ be a group acting on a space $X$. Show that, for each $n$, there is an induced group action of $G$ on $H_n(X)$.

38. (a) Let $A \subseteq X$. Prove or find a counterexample: for each $n$,

$$H_n(X) \cong H_n(A) \oplus H_n(X, A).$$

(b) Let $A \subseteq X$ and suppose $A$ is a retract of $X$. Prove or find a counterexample: for each $n$,

$$H_n(X) \cong H_n(A) \oplus H_n(X, A).$$

39. Let $X$ be the quotient of the 2-sphere $X = S^2/\{a, b\}$ gluing together the two points $a$ and $b$. Let $p$ be the image of $\{a, b\}$ in $X$.
    (a) Compute the local homology groups $H_2(X, X \setminus \{p\})$ and $H_2(X, X \setminus \{x\})$ for $x \neq p$.
    (b) Prove that any homeomorphism of $X$ must fix the point $p$.

40. Let $X \subseteq K$ be a retract of $K$ by a retraction $r : K \to X$. Show that $r_* : H_n(K) \to H_n(X)$ is a projection onto a direct summand.

41. Prove or disprove the analogue of the hairy ball theorem for the torus.

42. (a) Prove that punctured $\mathbb{R}\mathrm{P}^n$ is homotopy equivalent to $\mathbb{R}\mathrm{P}^{n-1}$.
    (b) Use Mayer-Vietoris to give a new calculation of the homology of a non-orientable genus-$g$ surface $N_g = \#_g \mathbb{R}\mathrm{P}^2$.

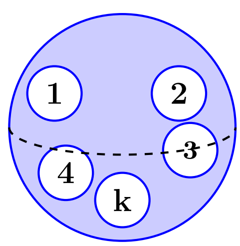
43. Compute the homology of $S^2 \times S^3$.

44. Must the following maps be nullhomotopic? Give a proof or prove a counterexample.
    (a) $f : S^2 \to S^1 \times S^1$
    (b) $g : S^1 \times S^1 \to S^2$

45. Let $X$ be the union of the $n$-sphere $S^n$ in $\mathbb{R}^{n+1}$ with the line segment of the $x_{n+1}$-axis connecting the north pole $(0, 0, \ldots, 0, 1)$ to the south pole $(0, 0, \ldots, 0, -1)$. Compute the fundamental group and homology groups of $X$.

46. Let $X$ be the space obtained from a torus and a Mobius strip by gluing the boundary circle of the Mobius band to the meridian circle of the torus. Compute the fundamental group and homology of $X$.

47. Consider the quotient map $q : S^n \to \bigvee_k S^n$ obtained by taking the quotient by the complement of $k$ disjoint embedded disks in $S^n$, as shown in the case $n = 2$.




Compute the map induced on cellular homology by $q$.

48. Find a CW complex structure on the 3-torus $S^1 \times S^1 \times S^1$, and use it to compute its homology.

49. A Hausdorff space $X$ is a *homology manifold of dimension* $n$ if for every $x \in X$,

$$H_k(X, X \setminus \{x\}) = \begin{cases} \mathbb{Z}, & k = n \\ 0, & \text{otherwise} \end{cases}$$

Show that an $n$-dimensional manifold $M$ is a homology manifold of dimension $n$.

50. Use the Mayer–Vietoris sequence to re-compute the homology of the genus-$g$ surface $\Sigma_g$, using the definition of $\Sigma_g$ as the connected sum $\Sigma_{g-1} \# \Sigma_1$.
*Hint:* First explain why a punctured genus-$h$ surface is homotopy equivalent to a wedge of $2h$ circles.

51. The infinite earring $E$ is a compact space such that $H_1(E)$ is not finitely generated (in fact, it is not even countably generated). Prove, in contrast, that if $X$ is a compact CW complex, its homology $\bigoplus_k H_k(X)$ is finitely generated.

52. Let $\Sigma_g$ be a closed orientable surface of genus $g$. For which $g$ is there a nontrivial covering map $p : \Sigma_g \to \Sigma_g$? Prove your answer.

53. Let $X$ and $Y$ be finite CW complexes. Derive a formula for $\chi(X \vee Y)$ in terms of $\chi(X)$ and $\chi(Y)$.

54. Let $N_g$ denote a non-orientable surface of genus $g$. Compute the homology of $N_g$ with coefficients $\mathbb{Z}/2\mathbb{Z}$.

55. Let $S^n$ denote a sphere of dimension $n \geq 0$. Let $G$ be an abelian group. Compute the homology groups $H_i(S^n; G)$ ...
    (a) ... using the long exact sequence of a pair.
    (b) ... using Mayer–Vietoris.

    You may use without proof that the natural analogues of both these long exact sequences hold for homology with coefficients.

56. Let $P_n$ be a regular $(2n)$-gon with parallel edges identified by a translation. Classify the surface $P_n$.

57. Let $X$ be the surface constructed from $\Sigma_g$ by deleting an open disk and then gluing in a Mobius band along its boundary. Classify the surface $X$.

58. **True or counterexample.** For each of the following statements: if the statement is true, write "True". If not, state a counterexample. No justification necessary.
*Note:* If the statement is false, you can receive partial credit for writing "False" without a counterexample.
    (i) Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of spaces. If one of $f$ or $g$ is nullhomotopic, then $g \circ f$ is nullhomotopic.
    (ii) Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of spaces. If $g \circ f$ is nullhomotopic, then one of $f$ or $g$ must be nullhomotopic.
    (iii) Let $F_t : X \to X$ be a homotopy of maps $F_0, F_1 : X \to X$. Let $A \subseteq X$. If $F_t(A) \subseteq A$ for all $t$, then $F_t$ induces a well-defined homotopy of functions $X/A \to X/A$.
    (iv) If $A$ is deformation retract of $X$, then $A$ and $X$ are homeomorphic.
    (v) Let $X$ be a finite CW complex. If $X$ is simply connected, then $X$ is contractible.
    (vi) Every map from $\mathbb{R}\mathrm{P}^1$ to $\mathbb{C}\mathrm{P}^1$ is nullhomotopic.
    (vii) Let $U, V$ be open subsets of a space $X$. If $U$ and $V$ are path-connected, and $U \cup V$ is simply connected, then $U \cap V$ is connected.
    (viii) Let $p : \tilde{X} \to X$ be a covering map. Then $p_* : H_n(\tilde{X}) \to H_n(X)$ is injective for every $n$.
    (ix) Let $B \subseteq A \subseteq X$. If $A$ is a deformation retract of $B$, then $H_n(X, A) \cong H_n(X, B)$ for all $n$.
    (x) If $f : X \to Y$ is a homotopy equivalence, then $f$ must surject
    (xi) If $f : S^n \to S^n$ is a homotopy equivalence, then $f$ must surject
    (xii) If a map of spheres $f : S^n \to S^n$ has no fixed points, then $f$ has degree $(-1)^{n+1}$
    (xiii) For any $n \geq 1$, there exists a map $f : S^n \to S^n$ of every degree $d \in \mathbb{Z}$.
    (xiv) If a continuous map $f : S^n \to S^n$ is a local homeomorphism at a point $x \in S^n$, then the local degree of $f$ at $x$ must be $\pm 1$.

(xv) Let $n \in \mathbb{Z}_{\geq 1}$. If $p : S^n \to X$ is a finite-sheeted cover, then it must be 1- or 2-sheeted.

(xvi) If the antipodal map $S^n \to S^n$ is homotopic to the identity map, then $n$ must be odd.

(xvii) If a space $X$ is simply connected, then $\widetilde{H}_0(X) = \widetilde{H}_1(X) = 0$.

(xviii) If $\widetilde{H}_0(X) = \widetilde{H}_1(X) = 0$, then $X$ is simply connected.

(xix) Let $X$ be a space and $A$ a subspace. If $A$ is contractible, then $H_n(X, A) = \tilde{H}_n(X)$ for all $n$.

(xx) Let $X$ be a space and $A$ a subspace. If $A$ is contractible, then $\tilde{H}_n(X/A) = \tilde{H}_n(X)$ for all $n$.

(xxi) If $X$ is a CW complex of dimension $n$, then $H_{n+1}(X) = 0$.

(xxii) If $X$ is a CW complex of dimension $n$, then $H_n(X) \neq 0$.

(xxiii) If $X$ is a CW complex of dimension $n$, then $H_n(X)$ is free abelian.

(xxiv) If a CW complex $X$ has no cells of dimension $d$, then $H_d(X) = 0$.

(xxv) If a CW complex $X$ has $H_d(X) = 0$, then $X$ has no cells of dimension $d$.

(xxvi) If a CW complex has cells in only even dimensions, then $H_n(X) = C_n(X)$ for all $n$, where $C_n(X)$ denotes the cellular chains on $X$.

(xxvii) Let $X$ be a CW complex with $k$-skeleton $X^k$. Then for all $n$, $H_n(X^n, X^{n-1})$ is free abelian.

(xxviii) Let $X$ be a finite CW complex. If $\chi(X) = 1$, then $X$ is contractible.

(xxix) Let $X$ be a finite CW complex. Then every map $f : X \to X$ that is homotopic to the identity must have a fixed point.

(xxx) Let $X$ be a contractible compact manifold. Then every continuous map $f : X \to X$ has a fixed point.

(xxxi) Let $X$ be a contractible manifold. Then every continuous map $f : X \to X$ has a fixed point.

59. Explain the value in defining and studying the fundamental group $\pi_1(X)$ of a space $X$.

60. Explain the value in defining and studying the homology groups $H_*(X)$ of a space $X$.

61. Suppose that I am convinced that absolute homology groups $H_*(X)$ of a space $X$ are a useful homotopy invariant, but I do not know why we define relative homology groups $H_*(X, A)$. Explain the value of defining relative homology groups as a tool to prove results about absolute homology groups.

62. Write a bullet-point summary of all the major results we have proved about ...
    (a) fundamental group
    (b) covering spaces
    (c) homology

63. What tools do we have to compute the following for a given space $X$? How might we recognize which tool to try?
    (a) fundamental group
    (b) covering spaces
    (c) homology

64. Write a bullet-point summary of all the major results we have proved about the following spaces.

    You may wish to include:
    - their definition
    - CW complex structures, $\Delta$-complex structures,
    - whether they are compact, connected, path-connected, locally path-connected, semi-locally simply-connected, contractible

- $\pi_1$
- their universal covers, other covers
- their homology
- their Euler characteristics
- any other results, such as fixed point theorems or results on vector fields

(a) $\mathbb{R}^n$

(b) disks $D^n$

(c) spheres $S^n$

(d) $n$-tori $(S^1)^n$

(e) graphs

(f) (orientable or nonorientable) closed surfaces

(g) punctured surfaces

(h) projective spaces $\mathbb{RP}^n$ and $\mathbb{CP}^n$

65. (a) **(QR Exam, May 2020).** Which of the following groups are fundamental groups of compact surfaces without boundary? For those which are, classify the surface:

(i) $\langle a, b, c \mid abca^{-1}b^{-1}c \rangle$

(ii) $\langle a, b, c, d \mid abcda^{-1}b^{-1}c^{-1}d^{-1} \rangle$

(iii) $\langle a, b, c \mid abcb^{-1}a^{-1}c \rangle$.

*Remark from Jenny:* We have not developed any general tools for proving a group presentation does *not* define the fundamental group of a surface, so my take on this question is that you are only expected to give proofs for the groups that *are* surface groups in the cases where this follows from standard CW-complex arguments.

(b) **(QR Exam, Sep 2018).** Consider two disjoint squares $ABCD$, $EFGH$ in $\mathbb{R}^2$. Identify their sides as follows:

$$AD \text{ with } HG,$$
$$DC \text{ with } EH,$$
$$AB \text{ with } BC,$$
$$EF \text{ with } FG.$$

All identifications of sides are bijective linear, with the endpoints identified in the order given. Is the quotient space of the identification a compact surface (i.e. a compact topological 2-manifold)? If so, classify it.

(c) **(QR Exam, May 2018).** Let $X$ be the space obtained by removing the open square in $\mathbb{R}^2$ with vertices $(11), (12), (21), (22)$ from the closed square with vertices $(00), (03), (30), (33)$. Now let $X$ be the space obtained by identifying the following pairs of line segments, directions indicated, via affine bijective maps:

$$(00), (03) \text{ with } (21), (22),$$
$$(30), (33) \text{ with } (11), (12),$$
$$(00), (30) \text{ with } (22), (12),$$
$$(03), (33) \text{ with } (21), (11),$$

(i) Calculate $\pi_1(X)$.

(ii) Prove that $X$ is a compact surface, and classify it.

(d) **(QR Exam, Jan 2018).** Let $Z$ be a convex 10-gon in the plane with vertices $A_0, A_1, A_2, A_3, A_4, B_4, B_3, B_2, B_1, B_0$ appearing in this order on the boundary (oriented counter-clockwise). Let $X$ be the topological space obtained from $Z$ by gluing the line segments $A_0A_1$ with $B_2B_3$, $B_0B_1$ with $A_2A_3$, $A_1A_2$ with $B_1B_2$, $A_3A_4$ with $B_3B_4$, $A_0B_0$ with $B_4A_4$. All pairs of line segments are attached by linear maps with the vertices corresponding in the order listed (first to first, last to last).

(i) Calculate $\pi_1(X)$.

(ii) Classify the surface $X$.

(e) **(QR Exam, Jan 2017).**
Let $A_k = e^{2k\pi i/2n}$. Let $C_n$ be the convex hull of $\{A_k \mid k = 0, 1, \dots, 2n-1\}$ with the topology induced from $C$. Let $\sim$ be the smallest equivalence relation on $C_n$ such that $tA_k + (1-t)A_{k+1} \sim (1-t)A_{k+n} + tA_{k+n+1}$, for all $k \in \mathbb{Z}/2n$, $0 \le t \le 1$. Let $X_n = C_n/\sim$ with the quotient topology.

(i) Calculate $\pi_1(X_n)$.

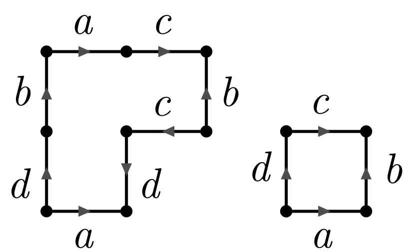
(ii) Classify the surface $X_n$

66. **(QR Exam, Jan 2022).** Suppose that a certain space $X$ decomposes as the union of three open subsets, $X = U_1 \cup U_2 \cup U_3$, satisfying the following properties.

- The open sets $U_1$, $U_2$, and $U_3$ are contractible.
- The pairwise intersections $U_1 \cap U_2$, $U_1 \cap U_3$, and $U_2 \cap U_3$ are contractible.
- The triple intersection $U_1 \cap U_2 \cap U_3$ is empty.

Prove that $X$ has the same homology as the circle $S^1$.

67. **(QR Exam, Aug 2021).** A space $X$ is constructed from two polygons with the following edge identifications. Compute the homology of $X$.




68. **(QR Exam, May 2021).** Let $X$ be the space obtained by glueing two copies of $S^3$ together along a (smoothly embedded) closed submanifold diffeomorphic to the torus $T = S^1 \times S^1$, i.e., $X = S^3 \cup_T S^3$. Calculate $H_*(X)$.

69. **(QR Exam, Jan 2021).** Let $G$ be a topological space admitting a topological group structure, i.e., one has a continuous multiplication map $\mu : G \times G \to G$ and a continuous inversion map $\iota : G \to G$ that define a group structure on the set $G$. Assume that $G$ is homeomorphic to a connected finite CW complex. Show that $\chi(G) = 0$ unless $G = \{1\}$.

70. **(QR Exam, May 2019).** Let $X$ be a 2-dimensional CW-complex with one 0-cell, four 1-cells $a, b, c, d$ and four 2-cells, attached along the loops $a^2bc, ab^2d, ac^2d, bcd^2$. Compute the homology of $X$.

71. **(QR Exam, Aug 2019).** Let a CW complex $X$ be obtained from a $k$-sphere, $k \ge 1$, by attaching two $(k+1)$-cells along attaching maps of degrees $m, n \in \mathbb{Z}$. Calculate the homology of $X$.

72. **(QR Exam, May 2019).** Let $S^1$ be the unit sphere in $\mathbb{C}$, let $T = S^1 \times S^1$ and let $T' = T/(S^1 \times \{1\})$. Let $X$ be the "connected sum" of $T$ and $T'$, i.e. a space obtained by cutting out interiors of closed 2-disks from $T$ and $T'$, respectively, (disjoint from the singular point in case of $T'$) and attaching the resulting spaces by the boundaries of the disks. Compute the fundamental group and homology of $X$.

73. **(QR Exam, May 2019).** For which values of $g \geq 0$ is it true that for every number $h \geq g$ ($g, h$ integers), a compact oriented surface $X$ of genus $g$ (without boundary) has a covering $f : Y \to X$ where $Y$ is a compact oriented surface of genus $h$?

74. **(QR Exam, Jan 2019).** Let $S_1, S_2$ be two disjoint copies of the $n$-sphere, $n > 1$ fixed. Choose two distinct points $A_i, B_i \in S_i$. Let $Z$ be a space obtained from $S_1 \sqcup S_2$ by identifying $A_1 \sim A_2$, $B_1 \sim B_2$. Compute, with proof, the lowest possible number of cells in a CW decomposition of $Z$.

75. **(QR Exam, May 2017).** Let $S^1$ be the set of complex numbers of absolute value 1 with the induced topology. $K$ be the quotient space formed from $S^1 \times [0, 1]$ by identifying every point $(z, 0)$ with the point $(z^{-2}, 1)$. Compute the homology of $K$.

76. **(QR Exam, May 2017).** Let $X$ be a connected CW-complex such that $H_i(X) = 0$ for all $i > 0$. Let $S^k$ denote the $k$-sphere. Prove that for all $k \in \mathbb{N}$, $H_n(X \times S^k)$ is $\mathbb{Z}$ for $n = 0$ and $n = k$, and 0 for all other values of $n$.

77. **(QR Exam, Sep 2016).** Let $Z = \{(x, y) \in \mathbb{C}^2 \mid x = 0 \text{ or } y = 0\}$. Find the homology of $\mathbb{C}^2 \setminus Z$ (with the subspace topology induced from the Euclidean topology on $\mathbb{C}^2$).

78. **(QR Exam, Jan 2016).** Let $U, V \subseteq S^n$, $n \geq 2$, be two non-empty connected open subsets such that $S^n = U \cup V$. Show that $U \cap V$ is connected.

79. **(QR Exam, Jan 2016).** Fix a prime number $p$. Let $X$ be a finite CW complex with an action of $G = \mathbb{Z}/p$.
    (a) If $\chi(X)$ is not divisible by $p$, show that the $G$ action on $X$ has a fixed point.
    (b) Give an example of such an action that is fixed-point free with $\chi(X) = 0$.

*Remark from Jenny:* I do not plan to test you on *homotopy pushouts* or their relationship to pushouts in the category of topological spaces. However, you can use the following problems to practice performing calculations with the Mayer–Vietoris long exact sequence, taking for granted that it can be used as we discussed in our second-to-last class.

80. **(QR Exam, May 2018).** Let $X$ be the pushout of the diagram

$$\begin{tikzcd} S^1 \times S^1 \arrow[r, "f"] \arrow[d, "g"'] & S^1 \\ S^1 & \end{tikzcd}$$

where $f$ is the projection to the first coordinate composed with a map of degree $k$, and $g$ is the projection to the first coordinate composed with a map of degree $\ell$. Compute $\pi_1(X)$.

81. **(QR Exam, Aug 2020).** Let $M$ be the Möbius band. Consider the pushout $X$ of

$$\begin{tikzcd} S^1 = \partial M \arrow[r] \arrow[d] & M \\ S^1 & \end{tikzcd}$$

where the horizontal map is the inclusion of the boundary, and the vertical map is a degree 2 covering space. Describe each $H_i(X)$ as an abelian group.

82. **(QR Exam, May 2018).** Let $f : S^2 \to S^2$ be a (continuous) map of degree $k$. Let $X$ be the pushout of the diagram

$$\begin{tikzcd} S^2 \arrow[r, "f"] \arrow[d, "\subset"'] & S^2 \\ D^3 & \end{tikzcd}$$

Calculate the homology of $X$.

83. **(QR Exam, Jan 2020).** For $n > 1$, let $X$ be the pushout of the diagram

$$\begin{tikzcd} \mathbb{RP}^{n-1} \arrow[r, "\subset"] \arrow[d, "\subset"'] & \mathbb{RP}^n \\ \mathbb{RP}^n & \end{tikzcd}$$

Compute the homology of $X$.

84. **(QR Exam, Jan 2018).** Let $f : S^2 \to S^2$ be a (continuous) map of degree $k$, and let $\pi : S^2 \to \mathbb{RP}^2$ be a covering. Let $X$ be the pushout of the diagram

$$\begin{tikzcd} S^2 \arrow[r, "\pi \circ f"] \arrow[d, "\subset"'] & \mathbb{RP}^2 \\ D^3 & \end{tikzcd}$$

Calculate the homology of $X$.

85. **(QR Exam, Jan 2017).** Let

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$
$$D^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$$
$$Q = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ or } z = 0 \text{ and } x^2 + y^2 \leq 1\}$$

Let $f : S^2 \to S^2$ be a (continuous) map of degree $k$. Let $X$ be the pushout of the diagram

$$\begin{tikzcd} S^2 \arrow[r, "\subset \circ f"] \arrow[d, "\subset"'] & Q \\ D^3 & \end{tikzcd}$$

Calculate the homology of $X$.