Terms and concepts covered: Homotopy; homotopic maps; nullhomotopic map. Homotopy rel a subspace. Homotopy equivalence; homotopy type; contractible. Deformation retraction. CW complex; weak topology. Products, wedge sums, and quotients of CW complexes.

Corresponding reading: Hatcher, Chapter 0, "Homotopy and homotopy type", "Cell complexes", "Operations on spaces" & Hatcher, Appendix, through Prop A.3.

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. **(Point-set review).**
   (a) Show by example that the inverse of a continuous, bijective function need not be continuous.
   (b) Prove that the continuous image of a connected set is connected.
   (c) Prove that the continuous image of a path-connected set is path-connected.
   (d) Prove that the continuous image of a compact set is compact.
   (e) Show by example that the preimage of a compact (respectively, connected, path-connected) under a continuous function need not be compact (respectively, connected, path-connected).
   (f) Show that a closed subset of a compact set is compact.
   (g) Show that, in a Hausdorff space, compact subsets are closed.
   (h) Show by example that compact sets need not be closed in general. *Hint:* consider finite topological spaces.
2. Let X be a topological space, and let $f, g : X \to \mathbb{R}^n$ be continuous maps. Show that f and g are homotopic via the homotopy
$$F_t(x) = t\, g(x) + (1-t) f(x).$$
3. Let X be a topological space. Show that all constant maps to X are homotopic if and only if X is path-connected. In general, what are the homotopy classes of constant maps in X?
4. Recall that a subset $S \subseteq \mathbb{R}^n$ is *star-shaped* if there is a point $x_0 \in S$ such that, for any $x \in S$, the line segment from x_0 to x is contained in S. Show that any star-shaped subset of $\mathbb{R}^n$ is contractible. Conclude in particular that convex subsets of $\mathbb{R}^n$ are contractible.
5. Show that a space X is contractible if and only if the identity map id_X is homotopic to a constant map $X \to X$.
6. Let X be a space and let $A \subseteq X$ be a deformation retract. Verify that X and A are homotopy equivalent.
7. Prove that every contractible space is path-connected. *Hint:* From a homotopy $F_t(x)$ we obtain, for each fixed x, a continuous function $t \mapsto F_t(x)$.
8. Find the mistake in the following "proof" that every path-connected space is contractible.

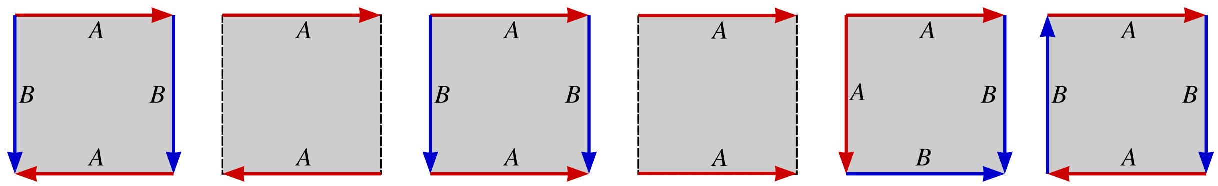
   False proof. Let X be a path-connected space, and choose a basepoint $x_0 \in X$. Then for each $x \in X$, there is some path $\gamma_x : [0,1] \to X$ from x to x_0. Define a homotopy
$$F_t(x) = \gamma_x(t).$$
   Then F_t is a homotopy from the identity id_X to the constant map at x_0. We conclude that X is contractible.
9. Let $S^1 \subseteq \mathbb{R}^2$ be the unit circle. Find the mistake in the following "proof" that S^1 is contractible.

False proof. There is a deformation retraction from S^1 to the point $(1,0)$ given by the homotopy

$$F_t(x,y) = \frac{(1-t)(x,y) + t(1,0)}{||(1-t)(x,y) + t(1,0)||}.$$

10. Suppose X and Y are homotopy equivalent spaces. Show that Y is path-connected if and only if X is.

11. **(Quotient surfaces).** Identify among the following quotient spaces: a cylinder, a Möbius band, a sphere, a torus, real projective space, and a Klein bottle.



12. Let $X = \bigcup_n X^n$ be a CW complex with n-skeleton X^n. Recall that we defined the topology on X so that a set U is open iff $U \cap X^n$ is open for every n.
    (a) Suppose that X is finite-dimensional, that is, $X = X^N$ for some N. Show that the topology on X agrees with the topology from our inductive definition of the N-skeleton X^N as a quotient space.
    (b) Again let X be any CW complex. Show that a set $C \subseteq X$ is closed iff $C \cap X^n$ is closed for every n.

13. We define a *subcomplex* of a CW complex X to be a closed subset that is equal to a union of cells. Show that a subcomplex A is itself a CW complex, by verifying inductively that the images of the attaching map of an n-cell in A must be contained in its $(n-1)$-skeleton A^{n-1}.

14. Verify that $\partial(D^n \times D^m) = (\partial D^n \times D^m) \cup (D^n \times \partial D^m)$. Draw pictures for some small values of m, n.

15. Verify the details of the natural CW complex structure on a product of CW complexes.

16. Verify the details of the natural CW complex structure on a quotient of a CW complex by a subcomplex.

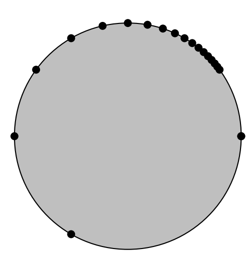
17. Let X be a CW complex. Show that the subspace topology on the 0-skeleton X^0 is discrete.

18. Let X be a CW complex, and X^n its n-skeleton.
    (a) The n-skeleton is defined by a quotient map $X^{n-1} \bigsqcup_\alpha D^n_\alpha \to X^n$. Show that the restriction of this map to X^{n-1} is injective.
    (b) Show that the quotient topology on X^{n-1} agrees with the subspace topology on its image in X^n.
    (c) Verify that the quotient topology on each skeleton agrees with their subspace topologies as subspaces of X.

19. Let X be a CW complex. See Assignment Problem 6 for the definition of the *characteristic maps* Φ_α. Verify that, for any α, the map Φ_α is continuous.

20. Recall that a CW complex is called *finite* if it has finitely many cells. Prove that any finite CW complex is compact, by realizing it as the continuous image of a finite union of closed balls.

21. Show that the following are NOT valid CW complex structures.
    (a) The closed interval I as an uncountable union of 0-cells.
    (b) A closed 2-disk, with one 2-cell and countably infinite 0- and 1-cells in its boundary.



Assignment questions

(Hand these questions in! Questions labelled "bonus" are optional.)

1. **(Homotopy defines an equivalence relation).**
   (a) Prove the following lemma.

   Lemma (Pasting Lemma). Let A, B be a topological spaces, and suppose A is the union $A = A_1 \cup A_2$ of closed subsets A_1 and A_2. Then a map $f : A \to B$ is continuous if and only if its restrictions $f|_{A_1}$ and $f|_{A_2}$ to A_1 and A_2, respectively, are continuous.

   (b) Let X, Y be topological spaces and consider the set of continuous maps $X \to Y$. Show that the relation "f is homotopic to g" defines an equivalence relation on this set.

2. **(Homotopy equivalence defines an equivalence relation).** Show that "homotopy equivalence" defines an equivalence relation on topological spaces.

3. **(Homotopies of maps of quotient spaces).**
   (a) State the definition of a *quotient map* and the *quotient topology*. State the universal property of a quotient map.

   (b) Prove or find a counterexample: The restriction of a quotient map to a subspace of its domain is a quotient map onto its image.

   (c) Suppose that X is a topological space with equivalence relation $\sim$, and let $X/\sim$ be the quotient space with the quotient topology. Let $I = [0, 1]$ be the unit interval with the Euclidean topology. Let

   $$q : X \times I \to (X/\sim) \times I$$

   be the natural map.
   Show that q is a quotient map, i.e., a set U in $(X/\sim) \times I$ is open if and only if $q^{-1}(U)$ is open. You can quote Munkres ''Elements of algebraic topology", Theorem 20.1, without proof.

   Theorem. Let $p : X \to (X/\sim)$ be a quotient map, and let C be a locally compact Hausdorff space. Then

   $$p \times id_C : X \times C \to (X/\sim) \times C$$

   is a quotient map.

   Deduce the following result about homotopies of maps of quotient spaces.

   Proposition. Suppose that X is a topological space with equivalence relation $\sim$, and let $X/\sim$ be the quotient space with the quotient topology. A homotopy $(X/\sim) \times I \to Y$ is continuous if and only if it arises from a continuous map $X \times I \to Y$ which is, for each fixed $t \in I$, constant on equivalence classes in X.

4. Consider the interval I as a CW complex with two vertices and one edge.
   (a) Let X be a CW complex. Describe the natural CW complex structure on $X \times I$. What is its n-skeleton, in terms of the skeleta of X?
   Remark: You can use without proof the result of Hatcher Theorem A.6, which implies that, because I is compact, the product topology agrees with the CW complex topology on $X \times I$.

   (b) Show that the n-skeleton of $X \times I$ is contained in $X^n \times I$. Use this to deduce that a homotopy $X \times I \to Y$ is continuous if and only if its restriction to $X^n \times I$ is continuous for every n.

   (c) **(S^∞ is contractible).** Define the infinite-dimensional sphere S^∞ as the space

   $$S^\infty = \bigcup_{n \geq 0} S^n = \left\{ (x_1, x_2, \ldots,) \;\middle|\; x_i \in \mathbb{R},\ x_i = 0 \text{ for all but finitely many } i, \sum_i x_i^2 = 1 \right\}$$

   It is topologized with the weak topology; a subset U is open if and only if $U \cap S^n$ is open for every n. Show that S^∞ is contractible. *Hints:*

- The map $S^\infty \to S^\infty$ given by $(x_1, x_2, x_3, \ldots) \longmapsto (0, x_1, x_2, x_3, \ldots)$ is continuous.
- Warm-up Problem 9.

5. **(A CW complex structure on the sphere).** Let S^n denote the n-sphere. In general we understand S^n is defined up to homeomorphism, but for the purposes of this question we will concretely define S^n to be the unit sphere in $\mathbb{R}^{n+1}$ with the Euclidean topology,
$$S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid |\mathbf{x}| = 1\}.$$
Let D^n denote the closed n-ball. Again, to be concrete we take D^n to be the unit ball in $\mathbb{R}^n$,
$$D^n = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| \leq 1\}.$$
   (a) Prove the following theorem.

   Theorem (A homeomorphism criterion). Let $f : X \to Y$ be a continuous, bijective map of topological spaces. Suppose X is compact and Y is Hausdorff. Then f is a homeomorphism.

   (b) Let $D^n/\sim$ be the quotient of D^n obtained by identifying all points in the boundary to a single point. Prove that $D^n/\sim$ is homeomorphic to S^n.

   Hint: Consider the map
$$f : D^n \longrightarrow S^n$$
$$x = (x_1, x_2, \ldots, x_n) \mapsto \begin{cases} \left(\left(2\sqrt{\frac{1}{||x||} - 1}\right) x_1, \ldots, \left(2\sqrt{\frac{1}{||x||} - 1}\right) x_n, 1 - 2||x||\right), \text{ if } x \neq 0 \\ \\ (0, 0, \ldots, 0, 1), \text{ if } x = 0 \end{cases}$$

   Remark: Going forward, you may assert without proof the identity of quotient spaces such as this one and the ones in Warm-Up Problems 11. But we should check this rigorously at least this once!

6. **Maps of CW complexes.**

   Definition (The characteristic map). Let X be a CW complex. For each n-cell e^n_α the associated *characteristic map* Φ_α is the composition
$$\Phi_\alpha : D^n_\alpha \hookrightarrow X^{n-1} \bigsqcup_\beta D^n_\beta \longrightarrow X^n \longrightarrow X$$
   Specifically $\Phi_\alpha|_{\partial D^n}$ is the attaching map ϕ_α, and Φ_α maps the interior of D^n homeomorphically to the (open) cell e^n_α. Its closure $\overline{e^n_\alpha}$ is called a *closed cell* of X, and we will see in Problem 8 that this closed cell is equal to the image $\Phi_\alpha(D^n_\alpha)$.

   (a) For a subset $A \subseteq X$, prove that A is open (respectively, closed) if and only if $\Phi_\alpha^{-1}(A)$ is open (respectively, closed) for every n and α. Deduce that the following map is a quotient map:
$$\sqcup \Phi_\alpha : \bigsqcup_{n,\alpha} D^n_\alpha \to X.$$
   (b) Deduce the following defining property of CW complexes.

   Proposition. Let X be a CW complex and Y a topological space. A map $f : X \to Y$ is continuous if and only if its restriction to every closed cell of X is continuous.

7. **(CW complexes are Hausdorff).**

   (a) Let X be a topological space. Recall that topologists say "points are closed" in X to mean that the singleton set $\{x\}$ is closed for all $x \in X$. Deduce from Problem 6 (a) that, in a CW complex, points are closed.

(b) Prove that a CW complex is Hausdorff. *Hint:* Read the first half of Hatcher p522, and explain the proof of Proposition A.3 in the special case that A and B are points. You may use the book as a reference while you write this proof, though you should not simply copy the book!

8. **(Compact subsets of CW complexes and the closure-finite property).**

(a) Let X be a CW complex, and let S be a (possibly infinite) subset of X such that every point of S is in a distinct cell of X. Prove that S is closed. Since the same argument applies to any subset of S, conclude that S has the discrete topology.

(b) Prove the following lemma.

Lemma (Compact subsets of CW complexes). Let X be a CW complex. Any compact subset of X intersects only finitely many (open) cells.

(c) Show that the closure of e^n_α in X is equal to the image of the characteristic map Φ_α.
Hint: The disk D^n is compact, and X is Hausdorff by Problem 7.

(d) "CW" stands for "closure-finiteness, weak topology". Prove the "closure-finiteness" property.

Proposition (Closure-finiteness of CW complexes). Let X be a CW complex. The closure of any (open) cell intersects only finitely many other (open) cells.

(e) The *infinite earring* is a subspace of $\mathbb{R}^2$ defined as the union $\bigcup_{n\geq 1} C_n$, where C_n is the circle of radius $\frac{1}{n}$ and center $(\frac{1}{n}, 0)$. See Figure 2. It is a favourite source of counterexamples in algebraic topology.[1]

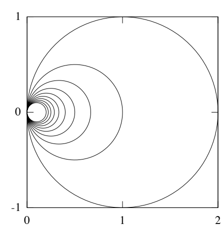


Figure 2: The infinite earring

Use part (b) to show that the topology on the infinite earring does not agree with the weak topology on a countable wedge of circles.
Remark: In fact, the infinite earring is not even homotopy equivalent to a CW complex.

9. **(Bonus: Homotopies as paths of maps).** Let X and Y be locally compact, Hausdorff topological spaces. Consider the space $C(X, Y)$ of continuous maps from X to Y with the compact-open topology. Let I be a closed interval. Show that the definition of a homotopy of maps $X \to Y$ is equivalent to the definition of a continuous map $I \to C(X, Y)$. In other words, a homotopy is a path through the space of continuous maps.

[1] The infinite earring is often called the *Hawaiian earring*, but there are concerns that this term is culturally insensitive, so I am trying to train myself to stop using it. Hatcher calls the space the *shrinking wedge of circles*. I do not love this term either, since, as you will prove, it is not homeomorphic to a wedge of circles.