Terms and concepts covered: Local degree and its relationship to degree, cellular homology.

Corresponding reading: Hatcher Ch 2.2, "Cellular Homology" (up to Example 2.36), "Mayer–Vietoris Sequences" (up to / including Example 2.46) .

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Consider the following short exact sequence of abelian groups,
$$0 \longrightarrow A \longrightarrow B \longrightarrow \mathbb{Z}^d \longrightarrow 0.$$
Verify that $B \cong A \oplus \mathbb{Z}^d$.

2. Consider the homomorphism $\mathbb{Z} \xrightarrow{m} \mathbb{Z}$ given by multiplication by $m \in \mathbb{Z}$.
   (a) Show that there cannot exist a homomorphism $r : \mathbb{Z} \to \mathbb{Z}$ satisfying $r \circ m = id_{\mathbb{Z}}$ unless $m = \pm 1$.
   (b) Suppose that $\iota : A \to X$ is the inclusion of a subspace A into a space X. Deduce that, if A is a retract of X, then $\mathbb{Z} \xrightarrow{m} \mathbb{Z}$ cannot be image of ι under a covariant functor $\underline{\text{Top}} \to \underline{\text{Ab}}$ unless $m = \pm 1$.

3. (a) Show that S^n has a Δ-complex structure defined inductively by gluing together two n-simplices Δ_1^n and Δ_2^n by the identity map on their boundaries.
   (b) Show that (with suitably chosen orientations) the corresponding simplicial homology group $H_n(S^n) \cong \mathbb{Z}$ is generated by the cycle $\Delta_1^n - \Delta_2^n$.
   (c) Compute the map induced on $H_n(S^n)$ by the reflection that fixes the equator S^{n-1} and interchanges the two simplices.

4. Suppose that $f : S^n \to S^n$ has no fixed points. Show that
$$f_t(x) = \frac{(1-t)f(x) - tx}{||(1-t)f(x) - tx||}$$
is a homotopy from f to the antipodal map $x \mapsto -x$. (Why does this homotopy require no fixed points?)

5. (a) Let $f : S^n \to S^n$ be a homeomorphism. Show that deg(f) must be ± 1.
   (b) Suppose that a continuous map $f : S^n \to S^n$ is not surjective, so f factors through a map
$$S^n \to S^n \setminus \{x\} \hookrightarrow S^n.$$
Show that $\deg(f) = 0$.
   (c) Show that, if $f \simeq g$, then $\deg(f) = \deg(g)$.
   (d) If $f, g : S^n \to S^n$, show that $\deg(f \circ g) = \deg(g)\deg(f)$.
   (e) Let $f : S^n \to S^n$ be a homotopy equivalence. Show that $\deg(f)$ must be ± 1.
   (f) Show that a reflection $S^n \to S^n$ has degree -1.
   (g) Show that the antipodal map $x \mapsto -x$ is the product of $(n+1)$ reflections. Conclude that it has degree $(-1)^{n+1}$.

6. Suppose that a continuous map $f : S^n \to S^n$ is not surjective. Show that f is nullhomotopic.

7. (a) Explain why any map $S^n \to S^n$ that factors $S^n \to D^n \to S^n$ must be degree zero.
   (b) Construct a surjective map $S^n \to S^n$ of degree zero.

8. Let $n \geq 1$. Explain why every map $S^n \to S^n$ can be homotoped to have a fixed point.

9. Let $x \in S^n$.
   (a) Describe a generator of $H_n(S^n, S^n \setminus \{x\})$.
   (b) Show that $H_n(S^n, S^n \setminus \{x\}) \cong H_n(U, U \setminus \{x\})$ for any neighbourhood U of x.
   (c) Let $f : S^n \to S^n$ be a continuous map. Let y be a point with a finite preimage $f^{-1}(y) = \{x_1, \ldots, x_m\}$. Let $U_1, \ldots U_m$ be small disjoint open balls around $x_1, x_2, \ldots, x_m$, respectively, that map to a small open ball V about y. Show that we can compute the local degree

$$f_* : H_n(U_i, U_i \setminus \{x_i\}) \longrightarrow H_n(V, V \setminus \{y\})$$

   by computing the degree

$$f_* : H_{n-1}(U_i \setminus \{x_i\}) \longrightarrow H_{n-1}(V \setminus \{y\})$$

   and give a topological description of the latter map.

10. We outlined proofs of the following facts about the homology of a CW complex X. Verify the facts directly in the case that the CW complex stuctue on X is a Δ-complex structure, by considering the simplicial homology groups.
   (a) If X is finite dimensional, $H_k(X) = 0$ for all $k > \dim(X)$.
   (b) More generally, for any Δ-complex X, $H_k(X^n) = 0$ for all $k > n$.
   (c) The inclusion $X^n \hookrightarrow X$ induces isomorrphisms $H_k(X^n) \xrightarrow{\cong} H_k(X)$ for all $k < n$.
   (d) The inclusion $X^n \hookrightarrow X$ induces a surjection $H_n(X^n) \twoheadrightarrow H_n(X)$.

11. Let X be a CW complex. Prove that the path-components of X are the path-components of its 1-skeleton X^1. Conclude that the map

$$H_0(X^k) \to H_0(X)$$

   induced by the inclusion of the k-skeleton is an isomorphism for all $k \geq 1$.

12. Let SX denote the suspension of a space X (Assignment Problem 3 (b)). Explain the homeomorphism $SS^n \cong S^{n+1}$ for all $n \geq 0$. (You do not need to check point-set details).
   In particular, Assignment Problem 3 (a) is a special case of Assignment Problem 3 (b).

Assignment questions

(Hand these questions in!)

1. **(Topology Qual, Sep 2017).** Prove that for positive integers n, k, there does not exist a covering $\pi : S^{2n} \to X$ where X is a simplicial complex with $\pi_1(X) \cong \mathbb{Z}/(2k+1)$.

2. **Mayer–Vietoris.**
   (a) Let X be a space, and let $A, B \subseteq X$ be subspaces whose interiors cover X. Let $C_n(A+B)$ denote the subgroup of the singular n-chain group $C_n(X)$ consisting of chains that are sums of a chain in A and a chain in B. Show that following is a short exact sequence of chain complexes.

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \longrightarrow 0$$

$$x \longmapsto (x, -x)$$

$$(y, z) \longmapsto y + z$$

(b) We will not prove this carefully, but it is possible to show (by subdividing simplices) that the inclusion of chain complexes
$$C_*(A+B) \to C_*(X)$$
induces isomorphisms on homology groups. Use this fact to deduce the following theorem, and describe the maps Φ and Ψ.

Theorem (The Mayer–Vietoris long exact sequence). Let X be a space, and let $A, B \subseteq X$ be subspaces whose interiors cover X. Then there is a long exact sequence on homology groups

$$\cdots \longrightarrow H_n(A\cap B) \xrightarrow{\Phi} H_n(A)\oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\delta} H_{n-1}(A\cap B) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_0(X) \longrightarrow 0.$$

Remark: By applying the same argument to the augmented singular chain complexes, we obtain a version of the Mayer–Vietoris long exact sequence for reduced homology groups (a result you don't need to check):

$$\cdots \longrightarrow \widetilde{H}_n(A\cap B) \xrightarrow{\Phi} \widetilde{H}_n(A)\oplus \widetilde{H}_n(B) \xrightarrow{\Psi} \widetilde{H}_n(X) \xrightarrow{\delta} \widetilde{H}_{n-1}(A\cap B) \longrightarrow \cdots$$
$$\cdots \longrightarrow \widetilde{H}_{-1}(X) \longrightarrow 0.$$

Some sources state this version only in the case that $A\cap B \neq \varnothing$, but the result holds in general as long as we use our calculation that $H_i(\varnothing)$ is $\mathbb{Z}$ in degree $i=-1$ and 0 for all other degrees i. All nonempty spaces X satisfy $\widetilde{H}_{-1}(X)\cong 0$.

(c) Verify the following statement from Hatcher (p150) about the connecting homomorphism δ. You do not need to verify the claim about barycentric subdivision.

"The boundary map $\delta : H_n(X) \to H_{n-1}(A\cap B)$ can easily be made explicit. A class $\alpha \in H_n(X)$ is represented by a cycle z, and by barycentric subdivision or some other method we can choose z to be a sum $x+y$ of chains in A and B, respectively. It need not be true that x and y are cycles individually, but $\partial x = -\partial y$ since $\partial(x+y)=0$, and the element $\delta\alpha \in H_{n-1}(A\cap B)$ is represented by the cycle $\partial x = -\partial y$, as is clear from the definition of the boundary map in the long exact sequence of homology groups associated to a short exact sequence of chain complexes."

3. **Some applications of Mayer–Vietoris.**

(a) Use the Mayer–Vietoris sequence to inductively re-compute the homology of S^n. *Hint:* Take A to be a neighbourhood of the top hemisphere, and B a neighbourhood of the bottom hemisphere.

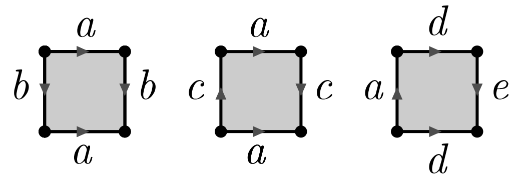
(b) **Definition (Suspension).** For a topological space X, the *(unreduced) suspension* SX of X is the quotient of $X\times I$ obtained by collapsing $X\times\{0\}$ to one point and collapsing $X\times\{1\}$ to another point.

Use the Mayer–Vietoris long exact sequence to prove that $\widetilde{H}_n(SX)\cong \widetilde{H}_{n-1}(X)$.
Hint: First explain why the images of $X\times[0,0.6)$ and $X\times(0.4,1]$ in SX are contractible.

(c) **(Topology Qual, Jan 2020).** The *unreduced suspension* $\tilde{X}$ of a space X is obtained from $X\times[0,1]$ by identifying $(x,0)\sim(y,0)$ and $(x,1)\sim(y,1)$ for all choices of points $x,y\in X$. If S^n is the n-sphere, $n>0$, compute the homology of the unreduced suspension of $S^n\times\{0,\ldots,k\}$.

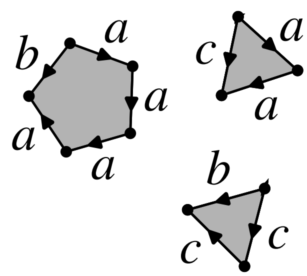
4. **Some applications of cellular homology.**

(a) **(Topology Qual, Jan 2022).** A space Y is constructed by gluing together a torus, a Klein bottle, and a cylinder along the edges labelled a below, i.e., Y is constructed from three squares using the edge identifications shown.



Calculate the homology of Y.

(b) **(Topology Qual, May 2024).** Let X be the quotient space defined as the union of the polygons below, modulo the given edge identifications.



(i) Compute the homology of X.

(ii) Let $B \subseteq X$ be the image of the loop b. Prove that B is not a retract of X.

5. **Some applications of the homology of $\mathbb{C}\mathrm{P}^n$ and $\mathbb{R}\mathrm{P}^n$.**

(a) State the conclusions of our calculations in class of the cellular homology of $\mathbb{C}\mathrm{P}^n$ and $\mathbb{R}\mathrm{P}^n$.

(b) **(Topology Qual, Jan 2021).** Let $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{C}\mathrm{P}^2$ be the natural map, sending a point $x \in \mathbb{C}^3 \setminus \{0\}$ to the line $\ell_x \in \mathbb{C}\mathrm{P}^2$ connecting x to 0 in $\mathbb{C}^3$. Does π admit a section (i.e., a right-inverse)?

(c) **(Topology Qual, Jan 2018).** Prove that every CW-structure on $\mathbb{R}\mathrm{P}^n$ has at least one cell in each dimension $0, 1, \ldots, n$.

(d) **(Topology Qual, Aug 2020).** Let $f : S^4 \to S^4$ be a map with the property that $f(x) = f(y)$ if y is the antipode of x. Show that $H_4(f) = 0$.