Terms and concepts covered: Categories; objects; morphisms; examples. Monic and epic morphisms. Covariant / contravariant functors. Universal property. Free groups: construction and universal property.

Corresponding reading: Hatcher Ch 0, "Cell complexes", "Homotopy extension property". Any reference on basic category theory (like Wikipedia or Tai-Danae Bradley's blog).

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Draw a collection of finite graphs (in the sense of graph theory). In each graph G, identify a maximal tree T. Verify that the quotient G/T is a wedge of 1-spheres. Use Assignment Problem 2 to explain why the quotient map $G \to G/T$ is a homotopy equivalence.

2. **(Monic and epic morphisms).**
   (a) Consider the category of sets, the category of abelian groups, and the category of topological spaces. Prove that in these categories, a morphism is monic if and only if it is a injective map.
   (b) Consider the category of sets, the category of abelian groups, and the category of topological spaces. Prove that in these categories, a morphism is epic if and only if it is a surjective map.
   (c) Prove that in the category of rings, the map $\mathbb{Z} \to \mathbb{Q}$ is an epic morphism that is not surjective.

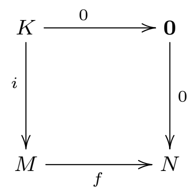
3. **Definition (Isomorphism).** Let $\mathscr{C}$ be a category. A morphism $f : X \to Y$ in $\mathscr{C}$ is an *isomorphism* if there exists a morphism $g : Y \to X$ in $\mathscr{C}$ such that $f \circ g = Id_Y$ and $g \circ f = Id_X$. Then we write $g = f^{-1}$, and we say that the objects X and Y are *isomorphic*.
   (a) Verify that this definition is agrees with your notion of "isomorphism" in every context you have encountered it.
   (b) Recall that the *homotopy category* <u>hTop</u> is the category of topological spaces and homotopy classes of continuous maps. Verify that an isomorphism in this category is precisely a homotopy equivalence.
   (c) Verify that "isomorphism" is an equivalence relation on objects in $\mathscr{C}$.
   (d) Let $\mathscr{C}$ be a category containing objects A and B, and let F be a functor $F : \mathscr{C} \to \mathscr{D}$. Show that if A and B are isomorphic objects of $\mathscr{C}$, then $F(A)$ and $F(B)$ will be isomorphic objects of $\mathscr{D}$.

4. **(Groups as categories).** Given a group G, define a category $\mathscr{G}$ with a single object ★ and morphisms $\mathrm{Hom}_{\mathscr{G}}(\bigstar, \bigstar) = \{g \mid g \in G\}$. The composition law is given by the group operation.
   (a) Show that a function between groups $G \to H$ is a group homomorphism if and only if the corresponding map between categories $\mathscr{G} \to \mathscr{H}$ is a functor.
   (b) (For those who have studied group representations). For a field k, let k–<u>vect</u> be the category of k-vector spaces and k-linear maps. Show that the definition of a functor from $\mathscr{G}$ to k–<u>vect</u> is equivalent to the definition of a linear representation of G over k.

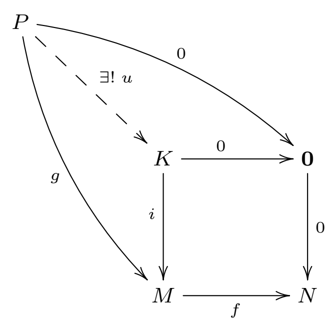
5. Let <u>Grp</u> be the category of groups. Consider the map Z : <u>Grp</u> $\to$ <u>Grp</u> that takes every group G to itself and every morphism f to the zero map. Is Z a functor?

6. (a) A *zero object* **0** in a category is an object with the following property: For any object M, there is a unique morphism from M to **0**, and a unique morphism from **0** to M. Show that if a category has a zero object, then it is unique up to unique isomorphism.
   (b) Let $\mathcal{C}$ be the category of abelian groups, and show that the zero module $\{0\}$ is a zero object. This definition allows us to define the *zero map* 0 between abelian groups M and N: it is the composition of the unique map $M \to \mathbf{0}$ with the unique map $\mathbf{0} \to N$.

(c) Let $\mathcal{C}$ be the category of abelian groups. If $f : M \to N$ is a morphism in $\mathcal{C}$, then define the *kernel* $i : K \to M$ of f to be the map i such that $f \circ i$ is the zero morphism 0



and satisfying the following: whenever there is a group homomorphism $g : P \to M$ such that $f \circ g = 0$, there is a unique map $u : P \to K$ such that $i \circ u = g$. In other words, there is a unique map u that makes the following diagram commute.



Verify that the inclusion of the kernel of an R–module map (in the way the kernel is usually defined, as the preimage of 0) does indeed satisfy this universal property.

(d) Show that this univeral property determines the map $i : K \to M$ uniquely up to unique isomorphism. Conclude that we therefore can indeed take this universal property as the *definition* of the kernel of f.

7. **(Power set functors).** Let $\underline{\mathrm{f\,Set}}$ denote the category of finite sets and all functions between sets. Let $\mathscr{P} : \underline{\mathrm{f\,Set}} \to \underline{\mathrm{f\,Set}}$ be the function that takes a finite set A to its *power set* $\mathscr{P}(A)$, the set of all subsets of A. If $f : A \to B$ is a function of finite sets, let $\mathscr{P}(f) : \mathscr{P}(A) \to \mathscr{P}(B)$ be the function that takes a subset $U \subseteq A$ to the subset $f(U) \subseteq B$.
   (a) Show that $\mathscr{P}$ is a covariant functor.
   (b) What if we had instead defined $\mathscr{P}(f) : \mathscr{P}(B) \to \mathscr{P}(A)$ to take a subset $U \subseteq B$ to its preimage $f^{-1}(U) \subseteq A$ under f?

8. **(Open subsets functor).** Let $\underline{\mathrm{Top}}$ be the category of topological spaces and continuous maps. Let $\underline{\mathrm{Set}}$ be the category of sets and all functions of sets. Define a *contravariant* functor $\mathcal{O} : \underline{\mathrm{Top}} \to \underline{\mathrm{Set}}$ that takes a topological space X to its collection $\mathcal{O}(X)$ of open subsets. How should we define $\mathcal{O}$ on morphisms to make it well-defined and functorial?

9. **(More adjoints).** Let $\underline{\mathrm{Top}}$ be the category of topological spaces and continuous maps. Let $\underline{\mathrm{Set}}$ be the category of sets and all functions of sets. Let $\mathcal{F}$ be the "forgetful map"

$$\mathcal{F} : \underline{\mathrm{Top}} \longrightarrow \underline{\mathrm{Set}}$$

that takes a space X to its underlying set. Define maps

$$I, D : \underline{\mathrm{Set}} \longrightarrow \underline{\mathrm{Top}}$$

so that for a set S, $D(S)$ is the set S with the discrete topology, and $I(S)$ is the set S with the indiscrete topology. Prove that there are bijections

$$\mathrm{Hom}_{\underline{\mathrm{Set}}}(A, \mathcal{F}(X)) \cong \mathrm{Hom}_{\underline{\mathrm{Top}}}(D(A), X)$$

and

$$\mathrm{Hom}_{\underline{\mathrm{Set}}}(\mathcal{F}(X), A) \cong \mathrm{Hom}_{\underline{\mathrm{Top}}}(X, I(A)).$$

It turns out that these bijections are "natural", so this result shows that D is a *left adjoint* to $\mathcal{F}$, and I is the *right adjoint* to $\mathcal{F}$.

Assignment questions

(Hand these questions in!)

1. **(Real and complex projective space: cell structures).**

   Definition (Real projective space). *Real n-dimensional projective space* $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Specifically, $\mathbb{RP}^n$ is the quotient space

   $$\mathbb{RP}^n \quad = \quad (\mathbb{R}^{n+1} \setminus \{0\}) \quad / \quad x \sim \lambda x \qquad \text{for any } \lambda \in \mathbb{R} \setminus \{0\}$$

   Definition (Complex projective space). *Complex n-dimensional projective space* $\mathbb{CP}^n$ is the space of 1-dimensional complex subspaces [1] through the origin in $\mathbb{C}^{n+1}$,

   $$\mathbb{CP}^n \quad = \quad (\mathbb{C}^{n+1} \setminus \{0\}) \quad / \quad x \sim \lambda x \qquad \text{for any } \lambda \in \mathbb{C} \setminus \{0\}$$

   (a) Show that $\mathbb{RP}^n$ is homeomorphic to the quotient of the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ via the equivalence relation $x \sim -x$, i.e., it is the quotient of S^n by the map that identifies antipodal points.

   (b) Illustrate $\mathbb{RP}^1$, $\mathbb{RP}^2$, and $\mathbb{CP}^1$ with pictures.

   (c) Read Hatcher Example 0.4 and 0.6 for a description of CW complex structures on $\mathbb{RP}^n$ and $\mathbb{CP}^n$. Summarize their construction here. (You may read the book as you write up your solution.)

2. The goal of this question is to prove the following theorem.

   Theorem (Collapsing contractible subcomplexes). Let X be a CW complex and $A \subseteq X$ a subcomplex. If A is contractible, then the quotient $X \to X/A$ is a homotopy equivalence.

   As an intermediate step, we will prove another important result: the homotopy extension property for CW subcomplexes. For this question, you may read Hatcher Chapter 0 "The Homotopy Extension Property" as you write up your solutions.

   (a) Define (qualitatively) a deformation retraction from $D^n \times I$ to $(D^n \times \{0\}) \cup (\partial D^n \times I)$.

   (b) Let X be a CW complex, and A a subcomplex. Hatcher writes,

   > "This deformation retraction [from part (a)] gives rise to a deformation retraction of $X^n \times I$ onto $(X^n \times \{0\}) \cup ((X^{n-1} \cup A^n) \times I)$, since $X^n \times I$ is obtained from $(X^n \times \{0\}) \cup ((X^{n-1} \cup A^n) \times I)$ by attaching copies of $D^n \times I$ along $(D^n \times \{0\}) \cup (\partial D^n \times I)$."

   Explain this construction in the case that X is the CW complex structure on the 2-disk shown in Figure 1, and A is the left edge. Illustrate (with pictures) the deformation retraction from $X^0 \times I$, $X^1 \times I$, and $X^2 \times I$.

   (c) Let A be a subcomplex of a CW complex X. Show that $(X \times \{0\}) \cup (A \times I)$ is a deformation retraction of $X \times I$. *Hint:* Perform the deformation retraction on $X^n \times I$ for the time interval $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$. See Hatcher Proposition 0.16. You do not need to provide point-set details.

[1] 1-dimensional complex subspaces are sometimes called *complex lines*, even though they are 2 real dimensional

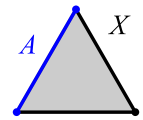
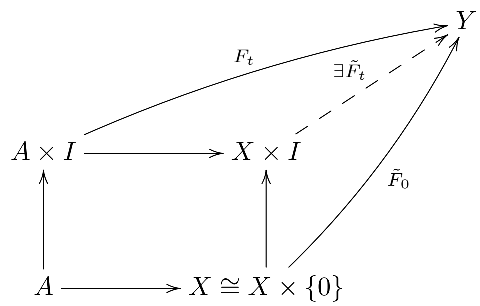


Figure 1: A CW complex structure on D^2

(d) **Definition (Homotopy extension property).** Let X be a topological space and $A \subseteq X$ a subspace. We say that the pair (X, A) has the *homotopy extension property* if, given a homotopy $F_t(a)$ from $A \times I \to Y$ and a map $\tilde{F}_0 : X \to Y$ such that $\tilde{F}_0|_A = F_0$, then there is a homotopy $\tilde{F}_t(x)$ from $X \to Y$ such that $\tilde{F}_t|_A = F_t$. The homotopy $\tilde{F}_t(x)$ is called an *extension* of $F_t(a)$.



The lift $\tilde{F}_t(x)$ need not be unique.

Let A be a subcomplex of a CW complex X. Show that (X, A) has the homotopy extension property. *Hint*: Use the pasting lemma, and the retraction defined by the deformation retraction from part (c) at time $t = 1$.

(e) Read Hatcher Proposition 0.17, which proves our theorem. Explain the steps in the proof and give explicit (if qualitative) descriptions of possibilities for the maps in the case that X is the graph in Figure 2 and A is its central edge.

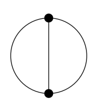
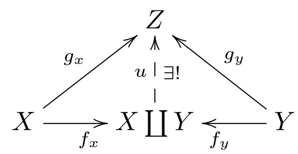


Figure 2: The theta graph

(f) Let X be a space and $A \subseteq X$ a subspace. The *cone* CA on A is the quotient space of $A \times I$ where $A \times \{1\}$ is collapsed to a point. Let Y be the space obtained by gluing CA to X by the identification $(a, 0) \sim a$ for all $a \in A$. Assuming that Y has a CW complex structure for which CA is a subcomplex, briefly explain why $Y \simeq X/A$.

(g) Our theorem does not hold for arbitrary contractible subspaces. Let $X = S^1$ and let A be the complement of a point in S^1, so A is homeomorphic to an open interval. Prove that S^1/A and S^1 are not homotopy equivalent, by proving S^1/A is contractible. (Next week we will prove that S^1 is not contractible).

3. **(Coproducts).** Let $\mathcal{C}$ be a category with objects X and Y. The *coproduct* of X and Y (if it exists) is an object $X \coprod Y$ in $\mathcal{C}$ with maps $f_x : X \to X \coprod Y$ and $f_y : Y \to X \coprod Y$ satisfying the following universal property: whenever there is an object Z with maps $g_x : X \to Z$ and $g_y : Y \to Z$, there exists a unique map $u : X \coprod Y \to Z$ that makes the following diagram commute:



(a) Let X and Y be objects in $\mathcal{C}$. Show that, if the coproduct $(X \coprod Y, f_x, f_y)$ exists in $\mathcal{C}$, then the universal property determines it uniquely up to unique isomorphism.

(b) Explain how to reinterpret this universal property as a bijection of sets

$$\operatorname{Hom}_{\mathcal{C}}\left(X \coprod Y, Z\right) \cong \operatorname{Hom}_{\mathcal{C}}(X, Z) \times \operatorname{Hom}_{\mathcal{C}}(Y, Z)$$

for objects X, Y, Z.

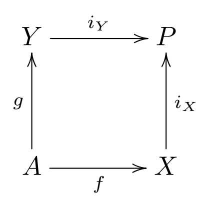
(c) Prove that in the category of sets, the coproduct $X \coprod Y$ of sets X and Y is their disjoint union.

(d) Let Top be the category of topological spaces and continuous maps. The coproduct of $X \coprod Y$ of spaces X and Y is called the *(topological) disjoint union*. The underlying set is the disjoint union. Describe the topology on the disjoint union and check that it satisfies the universal property.

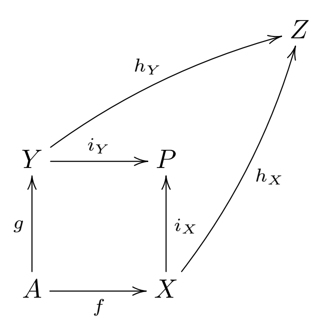
(e) Prove that in the category of abelian groups, the coproduct of groups $X \coprod Y$ is the direct sum $X \oplus Y$ with the canonical inclusions of X and Y. In other words, this universal property defines the direct sum operation on abelian groups.

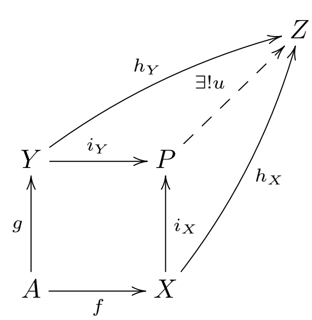
(f) In the category Grp of groups, the univeral property for the coproduct does *not* define the direct product operation. The coproduct $G \coprod H$ of groups G and H is a construction called the *free product* of G and H, and denoted $G * H$. Determine how to construct the group $G * H$ along with maps $G \to G * H$ and $H \to G * H$ that satisfy the universal property.

4. **(Pushouts).** Let $\mathcal{C}$ be a category with objects A, X, Y, and morphisms $f : A \to X$ and $g : A \to Y$. The *pushout* of f and g, if it exists, is an object P along with morphisms $i_X : X \to P$ and $i_Y : Y \to P$ making the following diagram commute,



and satisfying the following universal property. For any object Z and maps h_X, h_Y making the following diagram commute, there exists a unique morphism $u : P \to Z$ that makes the second diagram commute as shown.





If the pushout exists, the universal property defines the pushout uniquely up to unique isomorphism (we won't repeat this proof again for this question). The pushout P is often denoted $X \coprod_A Y$ or $X +_A Y$; it depends on the morphisms f and g but these are commonly suppressed from the notation.

(a) Suppose $\mathcal{C}$ is the category Set of sets and all functions. Describe the pushout of set maps $f : A \to X$ and $g : A \to Y$. No proof needed.

(b) Suppose $\mathcal{C}$ is the category Top of topological spaces and continuous maps. Describe the pushout of continuous maps $f : A \to X$ and $g : A \to Y$. No proof needed.

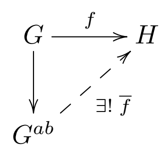
(c) Suppose $\mathcal{C}$ is the category Ab of abelian groups and group homomorphisms. Describe the pushout of group homomorphisms $f : A \to X$ and $g : A \to Y$. No proof needed.

5. **(Abelianization).** Let Grp denote the category of groups and group homomorphisms, and let Ab denote the category of abelian groups and group homomorphisms. Define the *abelianization* G^{ab} of a group G to be the quotient of G by its *commutator subgroup* $[G, G]$, the subgroup normally generated by *commutators*, elements of the form $ghg^{-1}h^{-1}$ for all $g, h \in G$.

   (a) Define a map of categories $[-,-] :$ Grp $\to$ Grp that takes a group G to its commutator subgroup $[G, G]$, and a group morphism $f : G \to H$ to its restriction to $[G, G]$. Check that this map is well defined (ie, check that $f([G, G]) \subseteq [H, H]$) and verify that $[-,-]$ is a functor.

   (b) Show that G^{ab} is an abelian group. Show moreover that if G is abelian, then $G = G^{ab}$.

   (c) Show that the quotient map $G \to G^{ab}$ satisfies the following universal property: Given any **abelian** group H and group homomorphism $f : G \to H$, there is a unique group homomorphism $\overline{f} : G^{ab} \to H$ that makes the following diagram commute:



   This universal property shows that G^{ab} is in a sense the "largest" abelian quotient of G.

   (d) Show that the map ab that takes a group G to its abelianization G^{ab} can be made into a functor $ab :$ Grp $\to$ Ab by explaining where it maps morphisms of groups $f : G \to H$, and verifying that it is functorial.

   (e) The category Ab is a subcategory of Grp. Define the functor $\mathcal{A} :$ Ab $\to$ Grp to be the inclusion of this subcategory; $\mathcal{A}$ takes abelian groups and group homomorphisms in Ab to the same abelian groups and the same group homomorphisms in Grp. Briefly explain why the universal property in Part (c) can be rephrased as follows: Given groups $G \in$ Grp and $H \in$ Ab, there is a natural bijection between the sets of morphisms:

$$\mathrm{Hom}_{\underline{\mathrm{Grp}}}(G, \mathcal{A}(H)) \cong \mathrm{Hom}_{\underline{\mathrm{Ab}}}(G^{ab}, H)$$

   Remark: Since this bijection is "natural" (a condition we won't formally define or check) it means that $\mathcal{A} :$ Ab $\to$ Grp and $ab :$ Grp $\to$ Ab are what we call a pair of *adjoint functors*.

6. **(Bonus)** Consider the rational numbers $\mathbb{Q}$ as a subspace of Euclidean space $\mathbb{R}$. Prove or disprove: $\mathbb{Q}$ does not have the homotopy type of a CW complex.

7. **(Bonus)** Prove or disprove: CW complexes are metrizable.