Terms and concepts covered: Path, homotopy of paths, composition of paths. Reparameterization. Loops, basepoint, fundamental group $\pi_1(X, x_0)$ of X based at x_0. π_1 of a product, π_1 as a functor.

Corresponding reading: Hatcher Ch 1.1, "Paths and Homotopy","Fundamental group of the circle", "Induced homomorphisms".

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. **(Constructing the free group).** In class, we constructed the free group F_S on a set S. Verify that our construction does indeed satisfy the universal property of the free group.

2. Let H be a group containing a subset S, and let $S \hookrightarrow H$ be the inclusion. In this question we investigate why, in general, H (along with the map $S \hookrightarrow H$) could fail to satisfy the universal property of the free group on S.
   (a) Suppose S generates H. Prove H satisfies the "uniqueness" condition of the universal property.
   (b) Show by example that, if S does not generate H, then H could fail to satisfy the "uniqueness" condition of the universal property.
   (c) Suppose that the elements of S satisfy some *relations* (a term we will define formally later in course). For example, the elements of S could commute, or might have finite order. Show that H will fail the "existence" condition of the universal property.

3. **(Free abelian groups).** Recall the universal property of the free group F_S on a set S: given any group G and any map of sets $f : S \to G$, the map f extends uniquely to a group homomorphism $\overline{f} : F(S) \to G$. In other words, there is a unique homomorphism $\overline{f}$ making the following diagram commute.

$$\begin{array}{ccc} S & \xrightarrow{f} & G \\ \downarrow & \nearrow_{\exists!\,\overline{f}} & \\ F_S & & \end{array}$$

   (a) Consider the same universal property in the category of abelian groups (so now G must be abelian). Show that the universal property defines the free abelian group on S, that is, $F_S \cong \bigoplus_S \mathbb{Z}$.
   (b) Why doesn't the free abelian group on S satisfy the universal property in the category of groups?

4. **Definition (Opposite category).** Let $\mathscr{C}$ be a category. The *opposite category* $\mathscr{C}^{op}$ is a category defined as follows: The objects of $\mathscr{C}^{op}$ are the same as the objects of $\mathscr{C}$. For objects $X, Y \in \mathscr{C}^{op}$, the morphisms are
$$\mathrm{Hom}_{\mathscr{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X).$$
   The composite $f \circ g$ of morphisms f, g in $\mathscr{C}^{op}$ is defined to be the morphism $g \circ f$ in $\mathscr{C}$.

   Informally, $\mathscr{C}^{op}$ is the category $\mathscr{C}$ after "reversing all the arrows". Show that the definition of a contravariant functor $\mathscr{C} \to \mathscr{D}$ is equivalent to the definition of a covariant functor $\mathscr{C}^{op} \to \mathscr{D}$.

5. Let $\mathscr{C}$ be a *locally small* category. ("Locally small" is a condition to deal with set-theoretic issues. All the categories we encounter will have this property). For each object $A \in \mathscr{C}$, we can define two *hom functors* from $\mathscr{C}$ to Set,
$$\mathrm{Hom}_{\mathscr{C}}(A, -) \qquad \text{and} \qquad \mathrm{Hom}_{\mathscr{C}}(-, A).$$

The first is covariant and the second is contravariant. They are defined as follows.

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(A,-) : \mathscr{C} &\longrightarrow \underline{\mathrm{Set}} \\ B &\longmapsto \mathrm{Hom}_{\mathscr{C}}(A,B) \\ [f : B \to C] &\longmapsto \left[f_* : \begin{array}{rl} \mathrm{Hom}_{\mathscr{C}}(A,B) & \to \mathrm{Hom}_{\mathscr{C}}(A,C)] \\ \phi & \mapsto f \circ \phi \end{array} \right] \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(-,A) : \mathscr{C} &\longrightarrow \underline{\mathrm{Set}} \\ B &\longmapsto \mathrm{Hom}_{\mathscr{C}}(B,A) \\ [f : B \to C] &\longmapsto \left[f^* : \begin{array}{rl} \mathrm{Hom}_{\mathscr{C}}(C,A) & \to \mathrm{Hom}_{\mathscr{C}}(B,A)] \\ \phi & \mapsto \phi \circ f \end{array} \right] \end{aligned}$$

(a) Verify that, for each object $A \in \mathscr{C}$, the maps $\mathrm{Hom}_{\mathscr{C}}(A,-)$ and $\mathrm{Hom}_{\mathscr{C}}(-,A)$ are functors.

(b) Explain the sense in that the forgetful functor $\underline{\mathrm{Top}} \to \underline{\mathrm{Set}}$ is "the same"* as the functor $\mathrm{Hom}_{\underline{\mathrm{Top}}}(\{*\},-)$. (*Technically, they are *naturally isomorphic* functors).

(c) Explain the sense in that the forgetful functor $\underline{\mathrm{Grp}} \to \underline{\mathrm{Set}}$ is "the same" as the functor $\mathrm{Hom}_{\underline{\mathrm{Grp}}}(\mathbb{Z},-)$.

(d) Fix a field k. Consider the forgetful functor k–$\underline{\mathrm{vect}} \to \underline{\mathrm{Set}}$. Is there is k-vector space V so that this functor is "the same" as $\mathrm{Hom}_{k-\underline{\mathrm{vect}}}(V,-)$?

If a functor $F : \mathscr{C} \to \underline{\mathrm{Set}}$ is naturally isomorphic to a hom functor, then F is called *representable*.

6. (a) Let $f : X \to Y$ and $g : Y \to X$ be maps of sets, and suppose that $f \circ g = id_Y$. Show that f is surjective, and g is injective.

   (b) Let $f : X \to Y$ and $g : Y \to X$ be morphisms in a category $\mathcal{C}$ such that $f \circ g = id_Y$. Show that f is an epimorphism, and g is a monomorphism.

   (c) Again let $f : X \to Y$ and $g : Y \to X$ be morphisms in a category $\mathcal{C}$ such that $f \circ g = id_Y$. Show moreover that the images of f and g under any covariant functor must also be an epimorphism and a monomorphism, respectively.

7. **(Homotopies of paths define an equivalence relation).** Let X be a space, and $x_0, x_1 \in X$. Consider all paths $\gamma : I \to X$ satisfying $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Show that the relation of being path homotopic (ie, homotopic rel $\{0,1\}$) is an equivalence relation on these paths.

8. **(Homotopy of paths respects composition of paths).**

   (a) Show that homotopy of paths is compatible with composition of paths. In other words, suppose we have points x_0, x_1, x_2 in a space X. Suppose that paths α and α' from x_0 to x_1 are homotopic rel $\{0,1\}$, and suppose that paths β and β' from x_1 to x_2 are homotopic rel $\{0,1\}$. Verify that the paths $\alpha \cdot \beta$ and $\alpha' \cdot \beta'$ from x_0 to x_2 are homotopic rel $\{0,1\}$.

   (b) What would happen if we just considered the paths α and β up to homotopy (instead of homotopy rel $\{0,1\}$)? Would homotopy still respect composition of paths?

9. **(Loop spaces).** For a topological space X with basepoint x_0, let ΩX denote the set of loops in X based at x_0. The loop space ΩX has a binary operation given by composition of loops. Explain why (in general) ΩX fails to be a group with this operation, by considering whether each of the associativity, identity, and inverse axioms will hold on the level of loops (in contrast to "loops up to path homotopy").

10. **(Paths in $\mathbb{R}^n$).**

    (a) Let $\gamma : I \to \mathbb{R}^n$ be a path from x_0 to x_1. Use the straight-line homotopy to show that γ is homotopic rel $\{0,1\}$ to any other path in $\mathbb{R}^n$ from x_0 to x_1.

    (b) Deduce that $\pi_1(\mathbb{R}^n, 0)$ is the trivial group.

11. In Assignment Problem 2 you will prove $\pi_1(S^1) \cong \mathbb{Z}$.
    (a) Let X be a contractible space. Show that $\pi_1(X)$ is the trivial group.
    (b) Conclude that the S^1, the torus, and in general the n-torus are not contractible, nor is any product of the form $S^1 \times Y$.

12. (a) Prove that a map of spaces
$$\begin{aligned} Z &\longrightarrow X \times Y \\ z &\longmapsto (f_X(z), f_Y(z)) \end{aligned}$$
is continuous if and only if the component maps $f_X : Z \to X$ and $f_Y : Z \to Y$ are continuous.
    (b) Describe the bijection between paths in $X \times Y$ and pairs of paths in X and in Y. Similarly, describe the bijection between homotopies of maps in $X \times Y$ and pairs of homotopies in X and in Y.
    (c) Check the details of our proof that
$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

13. (a) Given a pair of continuous maps $f_1 : Z_1 \to W_1$ and $f_2 : Z_2 \to W_2$, show that their product is continuous,
$$\begin{aligned} f_1 \times f_2 : Z_1 \times Z_2 &\longrightarrow W_1 \times W_2 \\ (z_1, z_2) &\longmapsto \big(f_1(z_1), f_2(z_2)\big) \end{aligned}$$
    (b) Given homotopy equivalences of spaces $X_1 \simeq Y_1$ and $X_2 \simeq Y_2$, show that there is a homotopy equivalence $X_1 \times X_2 \simeq Y_1 \times Y_2$.

14. (a) Suppose that $f : X \to Y$ is a homeomorphism. Show that, for any subset $A \subseteq X$, f induces a homeomorphism $f|_{X-A} : (X - A) \to (Y - f(A))$.
    (b) Show that $\mathbb{R}^1$ is not homeomorphic to $\mathbb{R}^n$ for any $n > 1$.
    Hint: Consider the path components of $\mathbb{R}^1 - \{0\}$.
    (c) Show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for any $n > 2$.
    Hint: Consider the fundamental group of $(\mathbb{R}^2 - \{0\}) \cong \mathbb{R} \times S^1$.

15. For a continuous map $f : X \to Y$, we defined the induced map on fundamental groups
$$\begin{aligned} f_* : \pi_1(X, x_0) &\longrightarrow \pi_1(Y, f(x_0)) \\ [\gamma] &\longmapsto [f \circ \gamma]. \end{aligned}$$
Complete our proof that π_1 is a functor by checking
    - f_* is well-defined on homotopy classes
    - f_* is a group homomorphism
    - $(id_X)_* = id_{\pi_1(X, x_0)}$
    - $(f \circ g)_* = (f_*) \circ (g_*)$

Assignment questions

(Hand these questions in!)

1. *Hint:* These results are proved in Hatcher Ch 1.1. You may read their proofs there, but then put the book away and write your solutions independently!
    (a) **(Reparameterization preserves homotopy class).**

> **Definition (Reparameterization).** Let $\gamma : I \to X$ be a path in a space X. A *reparameterization* of γ is a path $\gamma \circ \phi$ obtained by precomposing γ by a map $\phi : I \to I$ such that $\phi(0) = 0$ and $\phi(1) = 1$.

Show that γ and any reparameterization $\gamma \circ \phi$ are homotopic rel $\{0, 1\}$.

(b) **(The fundamental group is a group)**. For a space X with basepoint x_0, we defined the fundamental group $\pi_1(X, x_0)$ to be the group of loops in X based on x_0 up to path homotopy, under composition of paths. Complete our proof that this is a group, by verifying the following. Let c be the constant loop at x_0, and let $\gamma, \gamma_1, \gamma_2, \gamma_3$ be any loops based at x_0.

- Associativity: $\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$ is a reparameterization of $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3$.
- Identity: $\gamma \cdot c$ is a reparameterization of γ. (A similar argument shows $c \cdot \gamma \simeq \gamma$).
- Inverses: $\gamma \cdot \overline{\gamma} \simeq c$, where $\overline{\gamma}(t) = \gamma(1-t)$. (A similar argument shows $\overline{\gamma} \cdot \gamma \simeq c$).

(c) **($\pi_1(X)$ is well-defined for path-connected X)**. Prove the following.

> **Theorem (Change of basepoint).** Let X be a space, and let x_0 and x_1 be two points in X connected by a path h. Then the *change-of-basepoint map*
>
> $$\begin{aligned} \pi_1(X, x_1) &\longrightarrow \pi_1(X, x_0) \\ [\gamma] &\longmapsto [h \cdot \gamma \cdot \overline{h}] \end{aligned}$$
>
> is an isomorphism. Here, $\overline{h}$ is defined as the path $\overline{h}(s) = h(1-s)$.

Conclude that (up to isomorphism) the fundamental group of X does not depend on the choice of basepoint, only on the choice of path component of the basepoint. If X is path-connected, it now makes sense to refer to "the" fundamental group of X and write $\pi_1(X)$ for the abstract group.

2. The goal of this question is to prove this theorem.

> **Theorem (The fundamental group of S^1).** Let S^1 denote the unit circle in $\mathbb{R}^2$. There is an isomorphism
>
> $$\begin{aligned} \Phi : \mathbb{Z} &\longrightarrow \pi_1\Big(S^1, (1, 0)\Big) \\ n &\longmapsto [\omega_n : t \mapsto (\cos(2\pi n t), \sin(2\pi n t))]. \end{aligned}$$

Hint: Hatcher proves this result in Theorem 1.7, using an approach that is closely related but not identical to the one below. If you read Hatcher's proof, please put the book away as you write your own solutions.

(a) Verify that $\Phi(m + n)$ and $\Phi(m) \cdot \Phi(n)$ are homotopic, so Φ is a group homomorphism.

(b) **Definition (Covering map).** Let $p : E \to B$ be a continuous map of topological spaces. The map p is called a *covering map* if every point $b \in B$ has some neighbourhood U_b with the following property. The preimage $p^{-1}(U_b) \subseteq E$ is the union of disjoint open sets $\{V_{b,\alpha}\}$ in E such that for each α the restriction $p|_{V_{b,\alpha}}$ is a homeomorphism from $V_{b,\alpha}$ to U_b. In this case, E is called a *covering space* of B.

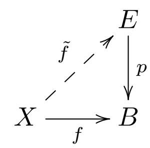
Prove that the map

$$\begin{aligned} p : \mathbb{R} &\longrightarrow S^1 \\ x &\longmapsto (\cos(2\pi x), \sin(2\pi x)) \end{aligned}$$

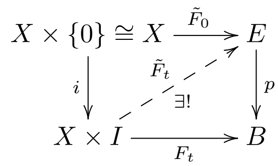
is a covering map.

(c) The following *homotopy lifting property* is a crucial feature of covering maps. We will prove it later in the course.

Definition (Lift). Let $p : E \to B$ be a covering map, and let $f : X \to B$ be a continuous map. A *lift* of f is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.



Theorem (Covering maps have the homotopy lifting property). Let $p : E \to B$ be a covering map, and let $F_t : X \times I \to B$ be a homotopy of maps $X \to B$. Then given any lift $\tilde{F}_0 : X \to E$ of F_0, there exists a unique lift $\tilde{F}_t : X \times I \to E$ of F_t whose restriction to $t = 0$ is the lift $\tilde{F}_0$.



Note that this theorem gives both *existence* and *uniqueness* of $\tilde{F}_t$. Briefly explain why the theorem implies the following two results.

(i) For each path $\gamma : I \to S^1$ starting at $(1, 0)$ and each $x \in p^{-1}(1, 0)$ there is a unique lift $\tilde{\gamma} : I \to \mathbb{R}$ starting at x .

(ii) Let $F_t : I \times I \to S^1$ be a homotopy rel $\{0, 1\}$ starting at $(1, 0) \in S^1$. For each $x \in p^{-1}(1, 0)$, there is a unique homotopy $\tilde{F}_t : I \times I \to \mathbb{R}$ with $\tilde{F}_0$ a path starting at $x \in \mathbb{R}$.

(d) Explain why the homotopy lifting property implies that the lifted homotopy $\tilde{F}_t$ in (ii) must be a homotopy rel $\{0, 1\}$. *Hint:* Consider the paths $t \mapsto F_t(0)$ and $t \mapsto F_t(1)$.

(e) Describe the path $\tilde{\omega}_n : I \to \mathbb{R}$ starting at $0 \in \mathbb{R}$ that lifts the loop

$$\begin{aligned} \omega_n : I &\longrightarrow S^1 \\ t &\longmapsto (\cos(2\pi n t), \sin(2\pi n t)), \end{aligned}$$

and describe the class of paths in $\mathbb{R}$ that are homotopic rel $\{0, 1\}$ to $\tilde{\omega}_n$.

(f) Prove that Φ is surjective and injective, hence an isomorphism.

3. In this question, we will develop some applications of our calculation $\pi_1(S^1) \cong \mathbb{Z}$.

(a) **Definition (Retraction).** Let X be a topological space, and $A \subseteq X$ a subspace. A *retraction* $r : X \to A$ is a continuous map such that $r(a) = a$ for all $a \in A$. The subspace A is called a *retract* of X.

(Note: a *deformation retraction* from X to A is a homotopy rel A from id_X to a retraction $r : X \to A$.) Suppose that $r : X \to A$ is a retraction. Let $\iota : A \to X$ denote the inclusion of A. Fix $a \in A$. Show that $\iota_* : \pi_1(A, a) \to \pi_1(X, a)$ is injective, and $r_* : \pi_1(X, a) \to \pi_1(A, a)$ is surjective.
Hint: Warm-up Questions 6 (a) and 15.

(b) Explain why no retraction from D^2 to $\partial D^2 = S^1$ can exist.

(c) Prove the following theorem.

Theorem (Brouwer fixed-point theorem for D^2). Let $f : D^2 \to D^2$ be a continuous map. Then D^2 has a *fixed point*, that is, there is some $x \in D^2$ such that $f(x) = x$.

Hint: Suppose $f : D^2 \to D^2$ has no fixed point. Use f to build a retraction from $r : D^2 \to S^1$. (Your map r should be constructed in a way that $r(x)$ depends continuously on the data of x and $f(x)$, but you do not need to prove that r is continuous).

(d) Recall that a *vector field* on D^2 is an ordered pair $(x, v(x))$ where $x \in D^2$ and $v(x)$ is a continuous map $v : D^2 \to \mathbb{R}^2$. We view $v(x)$ as a vector based at x. Prove the following theorem.

Theorem. Given a nonvanishing vector field on D^2, there exists a point $x \in S^1$ where the vector $v(x)$ points radially outward, and a point $y \in S^1$ where the vector $v(y)$ points radially inward.

Hint: Consider $F_t(x) = tx + (1-t)v|_{S^1}(x)$.

4. (a) Suppose $f_0, f_1 : X \to Y$ are homotopic maps via a homotopy f_t. Let $x_0 \in X$ be a basepoint, and let h be the path $h(t) = f_t(x_0)$. Prove that $\beta_h \circ (f_1)_* = (f_0)_*$, where β_h is the change-of-basepoint map,

$$\begin{aligned} \beta_h : \pi_1(Y, f_1(x_0)) &\longrightarrow \pi_1(Y, f_0(x_0)) \\ [\gamma] &\longmapsto [h \cdot \gamma \cdot \overline{h}] \end{aligned}$$

(b) Use (a) to deduce that if $f : X \to Y$ is nullhomotopic, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is the trivial map.

(c) Let $f, g : X \to Y$ be homotopic maps, and let

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \quad \text{and} \quad g_* : \pi_1(X, x_0) \to \pi_1(Y, g(x_0))$$

be their induced maps. Use part (a) to show that if f_* is injective, surjective, or trivial, then so is g_*.

(d) Prove the following.

Theorem (π_1 is a homotopy invariant). If $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.

5. **(Bonus).** The *(unreduced) suspension* SX of a space X is the quotient of $X \times I$ that collapses $X \times \{0\}$ to a point and collapses $X \times \{1\}$ to a distinct point. Suppose that Z is a contractible CW complex and that $X \subseteq Z$ is a CW subcomplex. Show that the quotient Z/X is homotopy equivalent to the suspension SX.