Terms and concepts covered: Seifert–Van Kampen Theorem, group presentations, π_1 of a graph, π_1 of a CW complex. Covering spaces, lifting properties of covering spaces.

Corresponding reading: Hatcher Ch 1.2 and 1.A (up to Proposition 1A.2), Ch 1.3.

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Show that the free product $*_\alpha G_\alpha$ of trivial groups G_α is trivial.

2. Let G be a group with presentation $\langle S \mid R \rangle$, and let H be any group. Show that a map $S \to H$ extends to a group homomorphism $G \to H$ if and only if the images of the generators S satisfy every relation in R.

3. Let G and H be groups with presentations $\langle S_G \mid R_G \rangle$ and $\langle S_H \mid R_H \rangle$, respectively. Verify the following. *Hint:* Use Warm-up Problem 2 to verify that the group described by the presentation satisfies the appropriate universal property.
   (a) $G^{ab} = \langle S_G \mid R_G \cup \{sts^{-1}t^{-1} | s, t \in S_G\} \rangle$
   (b) $G * H = \langle S_G \cup S_H \mid R_G \cup R_H \rangle$
   (c) $G \oplus H = \langle S_G \cup S_H \mid R_G \cup R_H \cup \{sts^{-1}t^{-1} | s \in S_G, t \in S_H\} \rangle$
   (d) Let $g : A \to G$, $h : A \to H$ be group homomorphisms.
   $G *_A H = \langle S_G \cup S_H \mid R_G \cup R_H \cup \{g(a)h(a)^{-1} \mid a \in A\} \rangle$

4. Let $\{(X_\alpha, x_\alpha)\}$ be a collection of based topological spaces. Suppose that for each α, there exists some open neighbourhood U_α of x_α that deformation retracts onto x_α. Let $\bigvee_\alpha X_\alpha$ be the wedge sum obtained by gluing together the points x_α to a single point x_0.
   (a) Verify the details of our proof-outline from class: $\pi_1\left(\bigvee_\alpha X_\alpha, x_0\right) = *_\alpha \pi_1(X_\alpha, x_\alpha)$.
   (b) **Bonus.** Suppose that the spaces X_α are CW complexes and the points x_α are any choices of basepoint. Explain why the spaces X_α must satisfy the nieghbourhood-deformation-retract condition. *Hint:* This is proven in Hatcher A.4 and A.5. You can assume this fact without proof in our class.

5. Consider the decomposition of S^1 into two open intervals $S^1 = A \cup B$ as shown in Figure 1. Use this example to show the Van Kampen theorem requires the hypothesis that $A \cap B$ is path-connected.

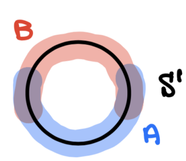


Figure 1: $S^1 = A \cup B$

6. (a) Describe how to construct the n-sphere S^n by gluing two n-disks D^n_A and D^n_B by their boundary along an $(n-1)$-sphere.
   (b) Assume $n \geq 2$. Decompose S^n into a union of open subsets $S^n = A \cup B$, where A is a neighbourhood of the image of D^n_A, and B is a neighbourhood of the image of D^n_B. See Figure 2. Use the Van Kampen theorem to show $\pi_1(S^n) = 0$.
   (c) Where does the proof go wrong when $n = 1$?

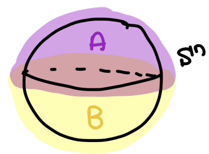


Figure 2: $S^n = A \cup B$

7. Find the error in the following flawed "proof" that the circle has trivial fundamental group.

   False proof. Decompose S^1 as a union of open intervals $S^1 = A \cup B \cup C$ as shown in Figure 3. Since A, B, C are open and path-connected, and their pairwise intersections $A \cap B$, $B \cap C$, $A \cap C$

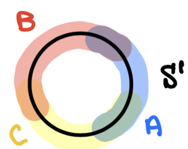


Figure 3: $S^1 = A \cup B$

   are path-connected, it follows that $\pi_1(S^1)$ is a quotient of the free product $\pi_1(A) * \pi_1(B) * \pi_1(C)$. Since A, B, C are contractible, $\pi_1(A) * \pi_1(B) * \pi_1(C)$ is trivial, so $\pi_1(S^1)$ is trivial.

8. (a) Let X be the complement of an open disk in the torus, as in Figure 4. Show that X deformation retracts onto the wedge $S^1 \bigvee S^1$. *Hint:* Consider the flat torus.

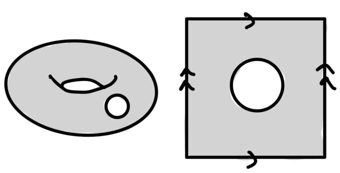


Figure 4: The torus with an open disk deleted

   (b) Deduce that $\pi_1(X)$ is the free group on 2 generators.

   (c) Apply the Van Kampen theorem to the decomposition of the torus T into open sets shown in Figure 5 to give an alternate computation of $\pi_1(T)$.

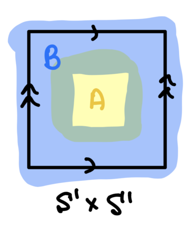


Figure 5: $T = A \cup B$

   (d) Apply the Van Kampen theorem to the decomposition of the closed genus-2 surface Σ_2 into open sets shown in Figure 6 to give an alternate computation of $\pi_1(\Sigma_2)$. Compare your solution to the result of Assignment Problem 2.

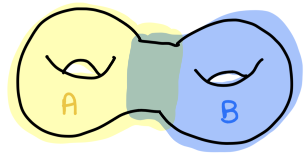


Figure 6: $\Sigma_2 = A \cup B$

9. Why, in the Van Kampen theorem, do we assume that each open set is path-connected?
10. What could go wrong, in the Van Kampen theorem, if we do not assume that each subset in our decomposition is open?
11. (a) Let G be a group, and suppose that G is generated by S. Show that the commutator subgroup of G is normally generated by the set

$$\{aba^{-1}b^{-1} \mid a, b \in S\}.$$

(b) Let G_1, G_2, H be groups, and let $f_1 : H \to G_1$ and $f_2 : H \to G_2$ be group homomorphisms. We defined the *free product with amalgamation* $G_1 *_H G_2$ to be the quotient of the free product $G_1 * G_2$ by the normal subgroup generated by the elements

$$\{f_1(h)f_2(h)^{-1} \mid h \in H\}.$$

Show that, if S is a generating set for H, then $G_1 *_H G_2$ is in fact the quotient by the normal subgroup generated by

$$\{f_1(s)f_2(s)^{-1} \mid s \in S\}.$$

12. Use the results of Assignment Question 1 to give a new proof that $\pi_1(S^n) = 0$ for all $n \geq 2$.
Hint: Use the CW complex structure on S^n where the 1-skeleton is a point.

Assignment questions

(Hand these questions in!)

1. **(The fundamental group of a CW complex).** In this question, we will continuing developing our program from class on computing the fundamental group of a CW complex.

(a) **Proposition (The effect of gluing in disks on π_1).** Let X be a path-connected space with basepoint x_0. Let Y be the space obtained from X by gluing in a number of disks D^2_α along their boundary by attaching maps $\varphi_\alpha : \partial D^2_\alpha \to X$. (By abuse of notation, we will also use φ_α to denote the loop $I \to X$ canonically determined by the map $\varphi_\alpha : S^1 \to X$.) For each α, let γ_α be a choice of path from x_0 to $\varphi_\alpha(1, 0)$. Then $\pi_1(Y, x_0)$ is the quotient of $\pi_1(X, x_0)$ by the subgroup normally generated by the loops $\{\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma_\alpha}\}_\alpha$.

This result is proven in Hatcher Proposition 1.26. Explain this proof (with pictures!) in the case that X is a 2-disk with 3 punctures, and Y is constructed by gluing two disks over two punctures via embedddings φ_1, φ_2, as shown in Figure 7. You may read Hatcher while you write your solution.

(b) **Definition (Cellular map).** A continuous map $f : X \to Y$ between CW complexes X and Y is called a *cellular map* if $f(X^n) \subseteq Y^n$ for all n.

The cellular approximation theorem is a major theorem in algebraic topology.

Theorem (Cellular approximation theorem). Every map $f : X \to Y$ of CW complexes is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subseteq X$, the homotopy may be taken to be stationary on A.

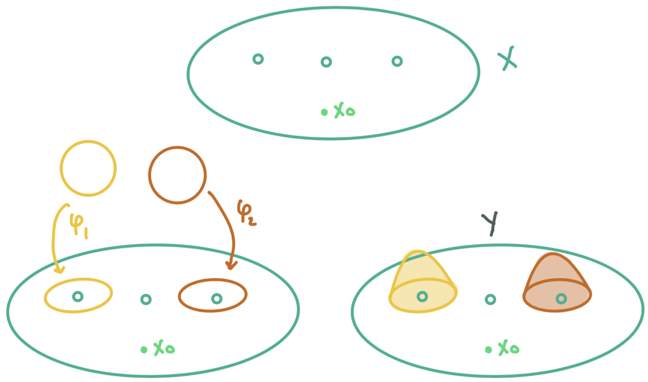


Figure 7: An instance of the spaces X and Y

Use the cellular approximation theorem to deduce the following theorem.

Theorem (The fundamental group of a CW complex is determined by its 2-skeleton). Let X be a path-connected CW complex. Let $\iota_1 : X^1 \to X$ and $\iota_2 : X^2 \to X$ be the inclusion of its 1-skeleton and 2-skeleton, respectively. The induced map $(\iota_1)_* : \pi_1(X^1) \to \pi_1(X)$ is surjective, and the induced map $(\iota_2)_* : \pi_1(X^2) \to \pi_1(X)$ is an isomorphism.

(c) In a few sentences, summarize our conclusions from class on how to compute a presentation for π_1 of a CW complex.

2. **(Surfaces of different genera are not homeomorphic, or even homotopy equivalent).**

Definition (Connected sum). Let M_1 and M_2 be n-manifolds. The *connected sum* $M_1 \# M_2$ is the n-manifold constructed as follows. Delete an open n-ball B_i from M_i. Let $h : \partial B_1 \to \partial B_2$ be a homeomorphism. Then glue $M_1 \setminus B_1$ to $M_2 \setminus B_2$ via h:

$$M_1 \# M_2 = (M_1 \setminus B_1) \cup (M_2 \setminus B_2) \quad / \quad x \sim h(x) \quad \text{for all } x \in \partial B_1$$

Fact: If M_1 and M_2 are oriented path-connected closed manifolds, then up to homeomorphism $M_1 \# M_2$ is independent of the choice of balls and (orientation-reversing) homeomorphism h.

Definition (Closed genus-g surface). The *(closed) genus-1 surface* Σ_1 is a torus. In general the *(closed) genus-g surface* Σ_g is the connected sum of g tori.

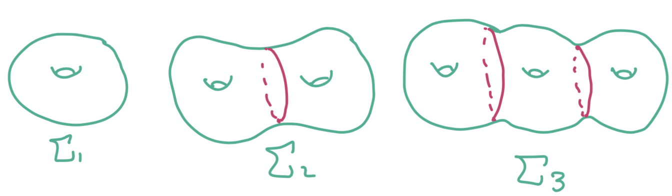


Figure 8: Surfaces of genus 1, 2, 3

(a) Briefly explain why the surface Σ_g can be realized as the quotient of a $4g$-gon by the edge identifications shown in Figure 9.
Hint: See Figure 10.

(b) Conclude that

$$\pi_1(\Sigma_g) = \langle a_1, b_1, a_2, b_2, \dots a_g, b_g \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \rangle \qquad \text{where } [a, b] := aba^{-1}b^{-1}.$$

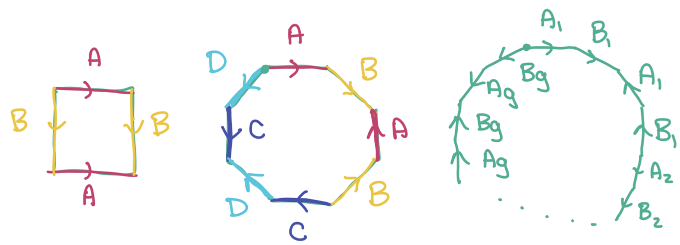


Figure 9: Σ_1, Σ_2, and Σ_g as quotients of polygons

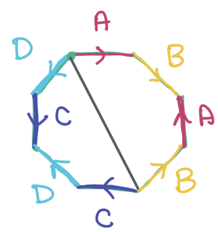


Figure 10

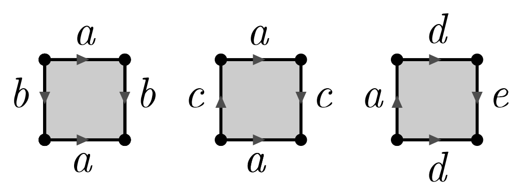
(c) Show that the abelianization of $\pi_1(\Sigma_g)$ is $\mathbb{Z}^{2g}$. Conclude that the surfaces Σ_g and Σ_h are not homotopy equivalent for any $g \neq h$.

3. In this problem we will apply Van Kampen and/or the results of Assignment Problem 1 to calculate some fundamental groups.

   (a) Compute the fundamental groups of $\mathbb{RP}^n$ and $\mathbb{CP}^n$ for all n. *Hint:* Use the CW complex structures you computed on Homework #2.

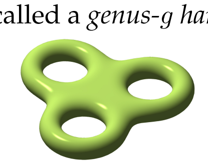
   (b) **(QR Exam, August 2019).** Let X be a space obtained from three copies of the Möbius strip by attaching their boundaries homeomorphically. Calculate $\pi_1(X)$ in terms of generators and defining relations.

   (c) **(QR Exam, January 2022).** A space Y is constructed by gluing together a torus, a Klein bottle, and a cylinder along the edges labelled a below, i.e., Y is constructed from three squares using the edge identifications shown.



   Calculate a presentation for the fundamental group of Y.

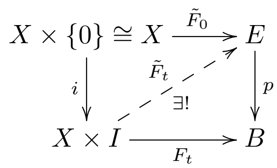
   (d) **(QR Exam, January 2024).** Fix $g \geq 0$. The closed orientable genus-g surface Σ_g is the boundary of a compact 3-dimensional manifold $\mathbf{H}_g$ called a *genus-g handlebody*, as pictured for $g = 3$.



   The *doubled handlebody* $\mathbf{D}_g$ is obtained by gluing two copies of $\mathbf{H}_g$ along their boundary via the identity map. Concretely, for $\mathbf{H} = \mathbf{H}' = \mathbf{H}_g$ and $I : \mathbf{H} \to \mathbf{H}'$ the the identity map, the space $\mathbf{D}_g$ is the quotient of the disjoint union $\mathbf{H}' \sqcup \mathbf{H}$ by the equivalence relation $I(x) \sim x$ for all $x \in \partial\mathbf{H} = \Sigma_g$. Compute $\pi_1(\mathbf{D}_g)$.

4. In this problem, we will prove that covering spaces satisfy the homotopy lifting property we encountered on Homework 2.

   Theorem (Covering maps have the homotopy lifting property). Let $p : E \to B$ be a covering map, and let $F_t : X \times I \to B$ be a homotopy of maps $X \to B$. Then given any lift $\tilde{F}_0 : X \to E$ of F_0, there exists a unique lift $\tilde{F}_t : X \times I \to E$ of F_t whose restriction to $t = 0$ is the lift $\tilde{F}_0$.



   Let $p : E \to B$ be a covering map, and let $F_t : X \times I \to B$ be a homotopy of maps $X \to B$. Let $\tilde{F}_0 : X \to E$ be a lift of F_0. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of B so that $p^{-1}(U_\alpha)$ can be decomposed as a disjoint union of open sets which are each mapped homeomorphically to U_α by p.
   Hint: For this proof, you may want to refer to the proof of Theorem 1.7 in Hatcher. Please put away the textbook as you write your solution.

   (a) We first address uniqueness. Suppose that $\gamma : I \to B$ is a path. Explain why there is a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of I so that for each i, $\gamma([t_i, t_{i+1}]) \subseteq U_i$ for some $U_i \in \mathcal{U}$.

   (b) Let $\tilde{b} \in p^{-1}(\gamma(0))$. Suppose that $\tilde{\gamma} : I \to E$ and $\tilde{\gamma}' : I \to E$ are two paths starting at $\tilde{b}$ lifting γ. Assume by induction that $\tilde{\gamma}|_{[0,t_i]} = \tilde{\gamma}'|_{[0,t_i]}$. Explain why necessarily $\tilde{\gamma}|_{[t_i,t_{i+1}]} = \tilde{\gamma}'|_{[t_i,t_{i+1}]}$. Conclude that there is a unique lift of γ starting at $\tilde{b}$.

   (c) Let $\tilde{F} : X \times I \to E$ be a homotopy lifting F_t and extending $\tilde{F}_0$. Explain why $\tilde{F}_t$ is unique.
   Hint: Consider $\tilde{F}|_{\{x\} \times I}$.

   (d) Now we address existence. Consider a point $x \in X$. Explain why there is a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of I (depending on x) and a neighbourhood $N_x \subseteq X$ of x such that, for each i, $F(N_x \times [t_i, t_{i+1}]) \subseteq U_i$ for some U_i in $\mathcal{U}$. Notably N_x is independent of i.
   Hint: First fix $x_0 \in X$ and consider neighbourhoods $N_t \times (a_t, b_t)$ of (x_0, t) for each $t \in I$. Use compactness of I.

   (e) Fix $x \in X$. Our next goal is to construct a lift $\tilde{F}^x$ of $F|_{N_x \times I}$ extending $\tilde{F}_0|_{N_x \times I}$. Assume by induction that $\tilde{F}^x$ is defined on $N_x \times [0, t_i]$. Describe how to extend $\tilde{F}^x$ to $N_x \times [0, t_{i+1}]$, and deduce that we can construct the desired lift $\tilde{F}^x$.
   Hint: You may need to replace N_x by a smaller neighbourhood of x.

   (f) For $x, y \in X$, explain why $\tilde{F}^x$ and $\tilde{F}^y$ must agree on $(N_x \cap N_y) \times I$. Explain how we can therefore combine the functions $\{\tilde{F}^x\}_{x \in X}$ to obtain a well-defined, continuous lift $\tilde{F}$ of the homotopy F extending $\tilde{F}_0$.

5. **(Bonus).** Read Chapter 2 Sections 2, 3, 4 of May's "Concise course in algebraic topology",

   `https://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf`

   (a) State and explain the definition of a groupoid.

   (b) State and explain the definition of a colimit. Show that coproducts and pushouts are special cases of colimits.

   (c) State and explain the groupoid version of the van Kampen theorem.

   (d) Use this theorem to give a new proof that $\pi_1(S^1) \cong \mathbb{Z}$.