Terms and concepts covered: Covering spaces, lifting properties of covering spaces.

Corresponding reading: Hatcher Ch 1.3

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Suppose that X is a connected space. Show that the only subsets of X that are both open and closed are X and $\varnothing$.

2. (a) Let $p : \tilde{X} \to X$ be a map and suppose an open subset $U \subseteq X$ is evenly covered by p (in the sense of Assignment Problem 1). Show that any open subset $V \subset U$ is also evenly covered by p.

   (b) Let $p : \tilde{X} \to X$ be a covering space. Deduce that every point of X has a neighbourhood basis of evenly covered open sets.

3. Let $p : \tilde{X} \to X$ be a covering map.
   (a) Let A be a subspace of X. Show that $p|_{p^{-1}(A)} : p^{-1}(A) \to A$ is a covering map.
   (b) Suppose B is a subspace of $\tilde{X}$. Is $p|_B : B \to p(B)$ necessarily a covering map?

4. Let $p : \tilde{X} \to X$ be a covering map. For $x_0 \in X$, show that $p^{-1}(x_0)$ is a topologically discrete set.

5. Some but not all sources require a covering map $p : \tilde{X} \to X$ to be surjective. Show that, even if p is not surjective, its image must be a union of connected components of X. *Hint:* Assignment Problem 2.

6. Prove that a covering map $p : \tilde{X} \to X$ is an open map, i.e., the image of an open subset is open.

7. **Definition (local homeomorphism).** A continuous map $f : X \to Y$ is a *local homeomorphism* if every point $x \in X$ has a neighbourhood U such that $f(U) \subseteq Y$ is open, and the restriction $f|_U : U \to f(U)$ is a homeomorphism.

   Definition (locally homeomorphic). A space X is *locally homeomorphic* to a space Y if every point in X has an open neighbourhood homeomorphic to an open subset of Y.

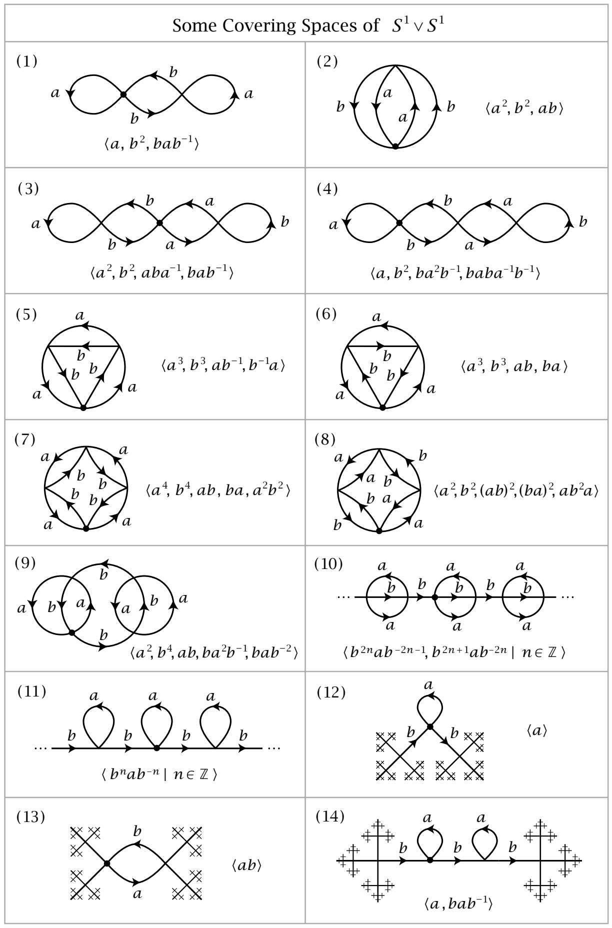
   Note that this definition is not symmetric in X and Y.
   (a) Show that if there exists a local homeomorphism $X \to Y$, then X is locally homeomorphic to Y. The converse is not true, for example, S^2 and $\mathbb{R}^2$ are locally homeomorphic to each other, but no local homeomorphism exists $S^2 \to \mathbb{R}^2$.
   (b) Verify that a covering map $p : \tilde{X} \to X$ is a local homeomorphism.
   (c) Verify that a local homeomorphism $f : X \to Y$ preserves local properties. For example, X will satisfy each of the following properties if and only if $f(X)$ does.
      (i) local connectedness and local path-connectedness
      (ii) local compactness
      (iii) first countability (every point has a countable neighbourhood basis)
      (iv) being locally Euclidean

8. Find an example of a continuous map that is a local homeomorphism but not a covering map.

9. Let $p : \tilde{X} \to X$ be a covering map. Show that, if X is Hausdorff, then so is $\tilde{X}$.

10. Consider the covers $p : \tilde{X} \to S^1 \vee S^1$ shown on Hatcher p58 (copied below).
    (a) For each cover, verify that it is a cover and that $p_*(\pi_1(\tilde{X}))$ is the subgroup shown.

(b) Consider the automorphism group of directed, labelled graphs $\tilde{X}$. This means the graph automorphisms that preserve the labels a and b and their orientations. For each cover $\tilde{X}$, determine whether this automorphism group acts transitively on vertices of $\tilde{X}$.



11. Show that S^n is a cover of $\mathbb{RP}^n$.

12. Prove that a 1-sheeted cover is precisely a homeomorphism.

13. (a) For each n, construct an n-sheeted cover of S^1.
    (b) Show that there is no 3-sheeted cover of $\mathbb{RP}^2$.
    Hint: Assignment Problem 2. What are the subgroups of $\pi_1(\mathbb{RP}^2)$?

14. Let $p : S^1 \to S^1$ be the cover $e^{i\theta} \mapsto e^{3i\theta}$ that "wraps" the circle around 3 times. Choose a basepoint x_0 in the base space and a lift $\tilde{x_0}$.
    (a) Show that the map p_* induced on fundamental groups is the map

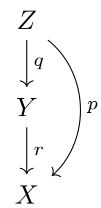
$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathbb{Z} \\ 1 &\longmapsto 3. \end{aligned}$$

    (b) Explicitly describe for this cover, with pictures, the map Φ from Assignment Problem 2. Verify that the map is well-defined on the cosets of the subgroup $H = 3\mathbb{Z}$ in $\mathbb{Z}$, and bijective.

Assignment questions

(Hand these questions in!)

1. Let p, q, r be continuous maps with $p = r \circ q$. Assume X is locally connected. Show that, if p and r are covering maps, then so is q.



   We will use this result later in our proof of the classification of covering spaces of a topological space.

   Hint: The following terminology may be convenient.[1] Let $p : \tilde{X} \to X$ be a continuous map of spaces. An open subset $U \subseteq X$ is called *evenly covered* by p if $p^{-1}(U)$ is the disjoint union of open subsets $\sqcup V_i \subseteq \tilde{X}$ such that, for each i, the restriction $p|_{V_i} : V_i \to U$ is a homeomorphism. We call the parts V_i of the partition $\sqcup V_i$ of $p^{-1}(U)$ *slices*. With this terminology, p is a covering map if and only if every point $x \in X$ has a neighbourhood which is evenly covered.

2. (a) Recall that a function N on a space X is *locally constant* if each $x \in X$ has a neighbourhood where N is constant. Show that a locally constant function is in fact constant on connected components of X.

   (b) **Definition (Sheets of a cover).** Let X be a connected space. The number of *sheets* of a cover $p : \tilde{X} \to X$ is the cardinality of $p^{-1}(x)$ for a point $x \in X$.

   Verify that the cardinality of $p^{-1}(x)$ is locally constant on X, and deduce that the number of sheets is well-defined for a cover of a connected space X.

   (c) Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a cover with $X, \tilde{X}$ path-connected. Consider the function

$$\begin{aligned} \phi : \pi_1(X, x_0) &\longrightarrow p^{-1}(x_0) \\ [\gamma] &\longmapsto \tilde{\gamma}(1) \end{aligned}$$

   where $\tilde{\gamma}$ is a lift of a representative γ starting at $\tilde{x_0}$. Show ϕ is well-defined.

   (d) Show moreover that ϕ is well-defined on right cosets of $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$, and so defines a function

$$\begin{aligned} \Phi : \pi_1(X, x_0) \text{ mod } H &\longrightarrow p^{-1}(x_0) \\ H[\gamma] &\longmapsto \tilde{\gamma}(1) \end{aligned}$$

   (e) Show that Φ is bijective. This proves the following theorem.

   Theorem (Sheets of a cover and π_1). Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a cover with $X, \tilde{X}$ path-connected. The number of sheets of p is equal to the index of $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ in $\pi_1(X, x_0)$.

   (f) The group $\mathbb{Z}^2$ has index-4 subgroups $(4\mathbb{Z} \times \mathbb{Z})$, $(\mathbb{Z} \times 4\mathbb{Z})$ and $(2\mathbb{Z} \times 2\mathbb{Z})$. Find a 4-sheeted covering map of the torus corresponding to each. No justification necessary.

3. (a) Suppose $p : \tilde{X} \to X$ is a covering map and that $\tilde{X}$ is compact. Show that p is finite-sheeted (on each component of X).

   (b) Let $p : \tilde{X} \to X$ be a surjective covering map that is finite-sheeted (on each component of X). Show that X is compact if and only if $\tilde{X}$ is.

[1] I do not think it is universally standard but it is used by Munkres.