Terms and concepts covered: lifting properties of covering spaces, classification of covering spaces

Corresponding reading: Hatcher Ch 1.3

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. **(Point-set review).** Let $f : X \to Y$ be a map of topological spaces. Verify that the following are equivalent.
   - f is continuous.
   - The preimage $f^{-1}(U)$ is open for every open subset $U \subseteq Y$.
   - The preimage $f^{-1}(C)$ is closed for every closed subset $C \subseteq Y$.
   - For every $x \in X$ and neighbourhood U of $f(x)$, there exists a neighbourhood V of x such that $f(V) \subseteq U$. [Note: If this condition holds for a particular point x, we say that f is *contitnuous at* x.]
   - Given a choice of (sub)basis for the topology on Y, $f^{-1}(B)$ is open for every (sub)basis element B.
   - For every subset $A \subseteq X$ there is containment $f(\overline{A}) \subseteq \overline{f(A)}$
   - (If X is first-countable, e.g. a metric space or a CW complex) f is sequentially continuous: If $(a_n)_{n\in\mathbb{N}}$ is a sequence of points in X converging to some limit a_∞, then $\big(f(a_n)\big)_{n\in\mathbb{N}}$ converges and its limit is $f(a_\infty)$.
2. (a) Give an example of a space that is simply connected but not contractible [we'll have the tools to prove this later in the course].
   (b) Prove that a graph is simply connected if and only if it is contractible (that is, if it is a tree).
3. See Assignment Problem 1 for the definitions of simply-connected, locally simply-connected, and semi-locally simply-connected.
   (a) Show that a simply-connected space is semi-locally simply-connected.
   (b) Show that a locally simply-connected space is semi-locally simply-connected.
   (c) Verify that S^1 is locally simply-connected and semi-locally simply-connected but not simply-connected.
   (d) Verify that the infinite earring is path-connected and locally path-connected but not semi-locally simply-connected.
   (e) Let CI be the cone on the infinite earring (in the sense of Homework #2 Problem 2(f)). Show that CI is simply-connected and semi-locally simply-connected, but not locally simply-connected.
   (f) Show that the topological disjoint union $CI \sqcup CI$ is semi-locally simply connected, but not simply connected or locally simply-connected.
4. (a) Explain how we can identify the universal cover of S^1 constructed in 1 with our cover $\mathbb{R} \to S^1$.
   (b) Describe a cover of S^1 associated to each subgroup of $\mathbb{Z}$.
5. (a) Explain how we can identify the universal cover of the torus constructed in 1 with $\mathbb{R}^2$.
   (b) Explain how to construct a cover of the torus from $\mathbb{R}^2$ for any subgroup of $\mathbb{Z}^2$.
6. A *torsion* group G is a group where every element has finite order. Show that the only group homomorphism from a torsion group G to a free abelian group $\mathbb{Z}^n$ is the zero map. What if G is generated by torsion elements?
7. Let $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ be covering maps, and let $f : X_1 \to X_2$ be an isomorphism of covers (Assignment Problem 5 (a)).

(a) Show that, for each $x \in X$, the map f defines a bijection between $p_1^{-1}(x)$ and $p_2^{-1}(x)$.

(b) Show that f^{-1} is also an isomorphism of covers.

(c) Verify that "isomorphism of covers" defines an equivalence relation on the covers of a fixed space X.

(d) Fix a cover $p : \tilde{X} \to X$. Show that the isomorphisms $\tilde{X} \to \tilde{X}$ form a group under composition.

(e) Compute the group of isomorphisms for the following covers.

- $\mathbb{R} \to S^1$
- The N-sheeted cover $S^1 \to S^1$
- $S^n \to \mathbb{R}\mathrm{P}^n$
- Your favourite covers from Hatcher's table on p58 (shown below).

8. Let $p : \tilde{X} \to X$ be the covering map of a connected cover, and let $\tau : \tilde{X} \to \tilde{X}$ be a deck transformation. Prove that if τ fixes a point, then τ is the identity map.

Assignment questions

(Hand these questions in!)

1. **(Construction of the universal cover).** Throughout this question, when we refer to "path homotopy", or "homotopy classes of paths", we implicitly mean homotopy rel $\{0, 1\}$.
Hint: You may read Hatcher p 63-65 while you complete this question.

Definition (simply-connected). A space X is called 0-*connected* if it is path-connected. The space is *simply-connected* or 1-*connected* if it is path-connected and $\pi_1(X) = 0$.

Definition (Locally simply-connected). A space X is *locally simply-connected* if each point $x \in X$ has a neighbourhood basis of simply-connected open sets U.

Definition (Semi-locally simply-connected). A space X is called *semi-locally simply-connected* if every point $x \in X$ has a neighbourhood U such that the inclusion $U \hookrightarrow X$ induces the trivial map $\pi_1(U, x) \to \pi_1(X, x)$.

Observe that in the definition of semi-locally simply-connected, the neighbourhood U does not need to be simply-connected. A loop in U based at x may not contract to the constant map at x by a path homotopy in U, but it does contract to the constant map by a path homotopy in the larger space X. See Warm-Up Problem 3.

Fact: CW complexes are locally contractible, therefore locally simply-connected and semi-locally simply-connected. Hatcher proves this in Proposition A.4 and in this course you may assume it without proof.

The goal of this problem is to construct a simply-connected cover of any path-connected, locally path-connected, semi-locally simply-connected space. We will later show that such a cover is (in an appropriate sense) unique, and it is called the *universal cover*.

(a) Suppose that X is simply-connected. Show that any two points in X are joined by a unique homotopy class of paths.

(b) Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering map. Use the lifting properties to show that there is a bijection between homotopy classes of paths in X starting in x_0 and homotopy classes of paths in $\tilde{X}$ starting at $\tilde{x_0}$.

(c) **Definition (The universal cover of X).** Let X be a path-connected, locally path-connected, semi-locally simply-connected space with basepoint x_0. Define

$$\tilde{X} = \{[\gamma] \mid \gamma \text{ a path in } X \text{ based at } x_0\}$$

where $[\gamma]$ is the homotopy class of the path γ. Define

$$\begin{aligned} p : \tilde{X} &\longrightarrow X \\ [\gamma] &\longmapsto \gamma(1) \end{aligned}$$

Let $\mathcal{U} = \{U \subseteq X \mid U \text{ is path-connected, open, and } \pi_1(U) \to \pi_1(X) \text{ is trivial}\}$. We topologize $\tilde{X}$ be defining a basis of open sets

$$U_{[\gamma]} = \{[\gamma\cdot\alpha] \mid U \in \mathcal{U}, \;\; \gamma \text{ a path from } x_0 \text{ to a point in } U, \;\; \alpha \text{ a path in } U \text{ with } \alpha(0) = \gamma(1)\ \}.$$

Show that the map p is well-defined and surjective.

(d) Show that $\mathcal{U}$ is a basis for the topology on X.

(e) Show that $U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$. Conclude that the sets $U_{[\gamma]}$ form the basis for a topology on $\tilde{X}$.

(f) Show that p is continuous. *Hint:* Show that, for $U \in \mathcal{U}$,

$$p^{-1}(U) = \bigcup_{\gamma \text{ a path from } x_0 \text{ to } U} U_{[\gamma]}.$$

(g) Show that $p|_{U_{[\gamma]}}$ is a homeomorphism to U. Deduce that p is a covering map.

(h) Show that $\tilde{X}$ is path-connected. *Hint:* For a point $[\gamma] \in \tilde{X}$, consider the path $t \mapsto [\gamma_t]$, where $[\gamma_t] \in \tilde{X}$ is the homotopy class of the path

$$\gamma_t(s) = \begin{cases} \gamma(s), & 0 \leq s \leq t \\ \text{constant function at } \gamma(t), & t \leq s \leq 1. \end{cases}$$

(i) Let $[x_0]$ denote the class of the constant path in X at x_0. Show that $\pi_1(\tilde{X}, [x_0]) = 0$.
Hint: We will prove in class that p_* is injective, so it suffices to show that its image is trivial. For a loop γ in X based at x_0, first show that $t \to [\gamma_t]$ is a lift of γ. Deduce that this lift is not a loop unless γ is trivial in $\pi_1(X, x_0)$.

(j) Give a brief/informal explanation of how we can identify the universal cover of $S^1 \vee S^1$ with the infinite tree shown in Figure 1.

2. **(The covering space of X associated to $H \subseteq \pi_1(X)$).** Let X be a path-connected, locally path-connected, semi-locally simply-connected space with basepoint x_0. Let $H \subseteq \pi_1(X, x_0)$ be a subgroup. Define X_H to be the quotient of the universal cover $\tilde{X}$ of X (Assignment Problem 1) by the equivalence relation

$$[\gamma] \sim [\gamma'] \quad \text{iff} \quad \gamma(1) = \gamma'(1) \text{ and } [\gamma \cdot \overline{\gamma'}] \in H.$$

Hint: You may read Hatcher Prop 1.36 while you complete this question.

(a) Verify that $\sim$ is well-defined and is an equivalence relation.

(b) Show that the covering map $p : \tilde{X} \to X$ factors through a map $p_H : X_H \to X$.

(c) Verify that p_H is a covering map.

(d) Show that the image of $(p_H)_*$ is H.

3. (a) The following lemma is proved in Hatcher Lemma 1A.3.

Lemma. Let X be a graph. Then every cover $\tilde{X}$ of X is a graph, with vertices and edges the lifts of vertices and edges, respectively, in X.

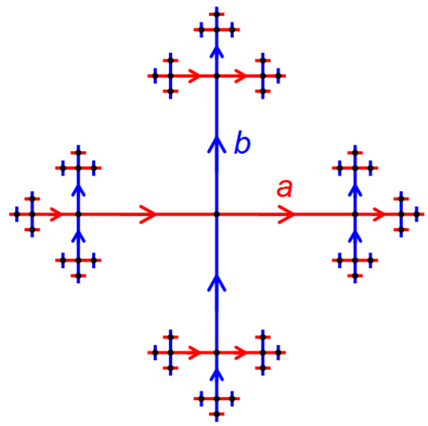

Figure 1: The universal cover of $S^1 \vee S^1$

Describe how to define a 1-dimensional CW complex structure on a cover $\tilde{X}$ of a graph X. You do not need to give point-set details. You may read Hatcher while you write your solution.

(b) Prove the following theorem.

Theorem. Every subgroup of a free group is free.

4. (a) **(Topology QR Exam, May 2016).** Let X be the complement of a point in the torus $S^1 \times S^1$.
   (i) Compute $\pi_1(X)$.
   (ii) Show that every map $\mathbb{R}\mathrm{P}^n \to X$ is nullhomotopic for $n \geq 2$.

   (b) **(Topology QR Exam, Jan 2022).** Let G be a graph, that is, a 1-dimensional CW complex. Let S^2 denote the 2-sphere. For each of the following statements, either prove the statement, or give (with justification) a counterexample.
   (i) Every continuous map $G \to S^2$ is nullhomotopic.
   (ii) Every continuous map $S^2 \to G$ is nullhomotopic.

   Some hints [which were not given on the QR exam]: (i) Cellular approximation theorem, (ii) you can assume the result of Warm-Up Problem 2 without proof.

5. **(The Classification of Covering Spaces).**

   (a) **Definition (Isomorphism of covers).** Let $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ be covering maps. A continuous map $f : X_1 \to X_2$ is an *isomorphism of covers* if f is a homeomorphism and $p_1 = p_2 \circ f$.

   Use the lifting properties and uniqueness of lifts proved in class to prove the following proposition.

   Proposition (Uniqueness of the cover associated to a subgroup of $\pi_1(X)$). If X is path-connected and locally path-connected, then two path-connected covering spaces $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ are isomorphic via an isomorphism $f : X_1 \to X_2$ taking a basepoint $\tilde{x_1} \in p_1^{-1}(x_0)$ to a basepoint $\tilde{x_2} \in p_2^{-1}(x_0)$ if and only if

$$(p_1)_*(\pi_1(X_1, \tilde{x_1})) = (p_2)_*(\pi_1(X_2, \tilde{x_2})).$$

   (b) Deduce the following important theorem, which is the culmination of your work on this and the previous assignment.

   Theorem (The classification of (based) covering spaces). Let X be path-connected, locally path-connected, and semi-locally simply-connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$, obtained by associating the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ to the covering space $(\tilde{X}, \tilde{x_0})$.

(c) Let $p : \tilde{X} \to X$ be a covering map with $\tilde{X}$ path-connected. Let $x_0 \in X$ and let $\tilde{x_0}, \tilde{x_1} \in p^{-1}(x_0)$. Analyze the change-of-basepoint map on $\tilde{X}$ to prove that $p_*(\pi_1(X, \tilde{x_0}))$ and $p_*(\pi_1(X, \tilde{x_1}))$ are conjugate subgroups of $\pi_1(X, x_0)$.

(d) Prove the following variation on the classification theorem.

> **Theorem (The classification of (unbased) covering spaces).** Let X be path-connected, locally path-connected, and semi-locally simply-connected. Then there is a bijection between the set of isomorphism classes of path-connected covering spaces $p : \tilde{X} \to X$ and the set of conjugacy classes of subgroups of $\pi_1(X)$.

6. **(Bonus).**

(a) For your choice of one of the problems on this assignment: after solving the problem and writing your solution up independently, try asking the problem to a generative AI chatbot like ChatGPT. Did it produce a correct answer? If not, what mistakes did it make? Were you able to coax it to arrive at the right answer? Comment on the quality of its solution.

(b) Have you found any ways (that do not violate any principles of academic integrity) to use generative AI in your work? Has it been useful? Discuss.