**Terms and concepts covered:** Deck transformations, regular covers.

**Corresponding reading:** Hatcher Ch 1.3

## Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $H$ be a subgroup of a group $G$.
   (a) Define the *normalizer* $N_G(H)$ of $H$ in $G$.
   (b) Show that $N_G(H)$ is a subgroup of $G$.
   (c) Show that $H$ is contained in $N_G(H)$, and is normal in $N_G(H)$.
   (d) Show that if $H$ is a normal subgroup of $G$, then $N_G(H) = G$.
   (e) Show that $N_G(H)$ is maximal in the following sense: if $J$ is a subgroup $H \subseteq J \subseteq G$ and $H$ is normal in $J$, then $J \subseteq N_G(H)$.

2. Let $X$ be path-connected, locally path-connected semi-locally simply-connected space. Explain why, if $X$ is simply connected, the only covers of $X$ are homeomorphisms $X \to X$.

3. Let $X, Y$ be path-connected, locally path-connected spaces. Assume $Y$ is semi-locally simply connected. Given a map $f : (X, x_0) \to (Y, y_0)$, which path-connected covers $\tilde{Y}$ of $Y$ will $f$ lift to?

4. We have illustrated a cover $\tilde{X}$ of $S^1 \vee S^1$ as a graph where every edge is directed and labelled by either $a$ or $b$. Explain how this convention encodes the data of the covering map, and explain why a deck transformation of $\tilde{X}$ is precisely a graph automorphism that preserves the direction and label of every edge.

5. (a) Suppose that $X$ has an abelian fundamental group. Explain why every connected cover of $X$ is regular.
   (b) Explain why every connected 2-sheeted cover is regular.

6. Let $G$ be a group acting on a set $X$.
   (a) Define *orbit*. Prove that the condition "$x_1$ and $x_2$ are in the same orbit" defines an equivalence relation on $X$.
   (b) State and prove the orbit-stabilizer theorem.

7. **Definition (Group acting on a space, I).** Let $G$ be a group. A group action of $G$ on a space $Y$ is a group homomorphism $\rho : G \to \mathrm{Homeo}(Y)$, where $\mathrm{Homeo}(Y)$ is the group of homeomorhpisms $Y \to Y$.

   *Note:* If $G$ were a topological group, we would want to impose extra conditions on our group action to be compatible with the topology. Here we assume that $G$ has no topology, or, equivalently, we may assume that $G$ has the discrete topology.

   Verify that the definition of a group action on a space is equivalent to the following.

   **Definition (Group acting on a space, II).** Let $G$ be a group. A group action of $G$ on a space $Y$ is a map

$$\begin{aligned} \alpha : G \times Y &\longrightarrow Y \\ (g, y) &\longmapsto g \cdot y \end{aligned}$$

   satisfying three conditions.

(i) For each fixed $g \in G$, the coresponding map is continuous:

$$\begin{aligned} g : Y &\longrightarrow Y \\ y &\longmapsto g \cdot y \end{aligned}$$

(ii) For each $g, h \in G$ and $y \in Y$, $(gh) \cdot y = g \cdot (h \cdot y)$.

(iii) For $e \in G$ the identity, $e \cdot y = y$ for all $y \in Y$.

8. (a) Let $G$ be a group with a covering space action on a space $Y$ (Assignment Problem 5). Prove that the action is free.

   (b) Show by example that a free action of a group $G$ on a space $Y$ need not be a covering action.
   *Hint:* $\mathbb{R}$ acts on $\mathbb{R}$.

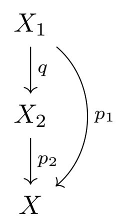
9. Let $p : Z \to W$ be a continuous surjective map. Show that, if $p$ is an open map, then it is a quotient map.

## Assignment questions

(Hand these questions in!)

1. **(The Galois correspondence for covering spaces).** In this question, we assume all spaces are path-connected, locally path-connected, and semi-locally simply-connected.

   (a) Suppose that $H_1, H_2$ are subgroups of the fundamental group $\pi_1(X, x_0)$ of a space $X$, and let $p_1 : (X_1, x_1) \to (X, x_0)$ and $p_2 : (X_2, x_2) \to (X, x_0)$ be the covering spaces such that $(p_1)_*$ and $(p_2)_*$ induce the inclusions of $H_1$ and $H_2$, respectively, into $\pi_1(X, x_0)$.
   Explain why $p_1$ factors through $p_2$ (as in the diagram below) if and only if $H_1 \subseteq H_2$.




   Conclude from Homework 5 Assignment Problem #1 that, if it factors, the map $q$ is a covering map.

   (b) Give a precise statement of the resulting strengthening of our classification theorem for based covering spaces of $X$: for every subgroup of $\pi_1(X, x_0)$ there is a unique covering space, and for every inclusion of subgroups $H_1 \to H_2$ there is an intermediate covering map.
   *Remark:* There is, in fact, an isomorphism of posets between the subgroups of $\pi_1(X)$ (ordered by inclusion) and the covers of $X$ (ordered by existence of intermediate covers). For a more detailed statement, see Hatcher Chapter 1.3 Problem 24. This result is sometimes called the *Galois correspondence* for covering spaces, in analogy to the Galois correspondence for field extensions.

   (c) Let $X = \mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2$. Draw the diagram of based covering maps of $X$ and intermediate covers, and label the fundamental group of each space.

2. **(The action of $\pi_1(X, x_0)$ on the fibres).** Let $X$ be a connected, locally path-connected, semi-locally simply-connected space. Let $p : \tilde{X} \to X$ be a covering map, and let $\alpha$ be a path in $X$. Define a map

$$L_\alpha : p^{-1}(\alpha(0)) \to p^{-1}(\alpha(1))$$

   as follows: for a point $\tilde{x_0} \in p^{-1}(\alpha(0))$, lift $\alpha$ to the path $\tilde{\alpha}$ starting at $\tilde{x_0}$. Then $L_\alpha(\tilde{x_0}) = \tilde{\alpha}(1)$.

   (a) Explain why $L_\alpha$ only depends on the homotopy class of $\alpha$ rel $\{0, 1\}$.

   (b) Show that $L_\alpha$ is a bijection of sets. *Hint:* What is its inverse?

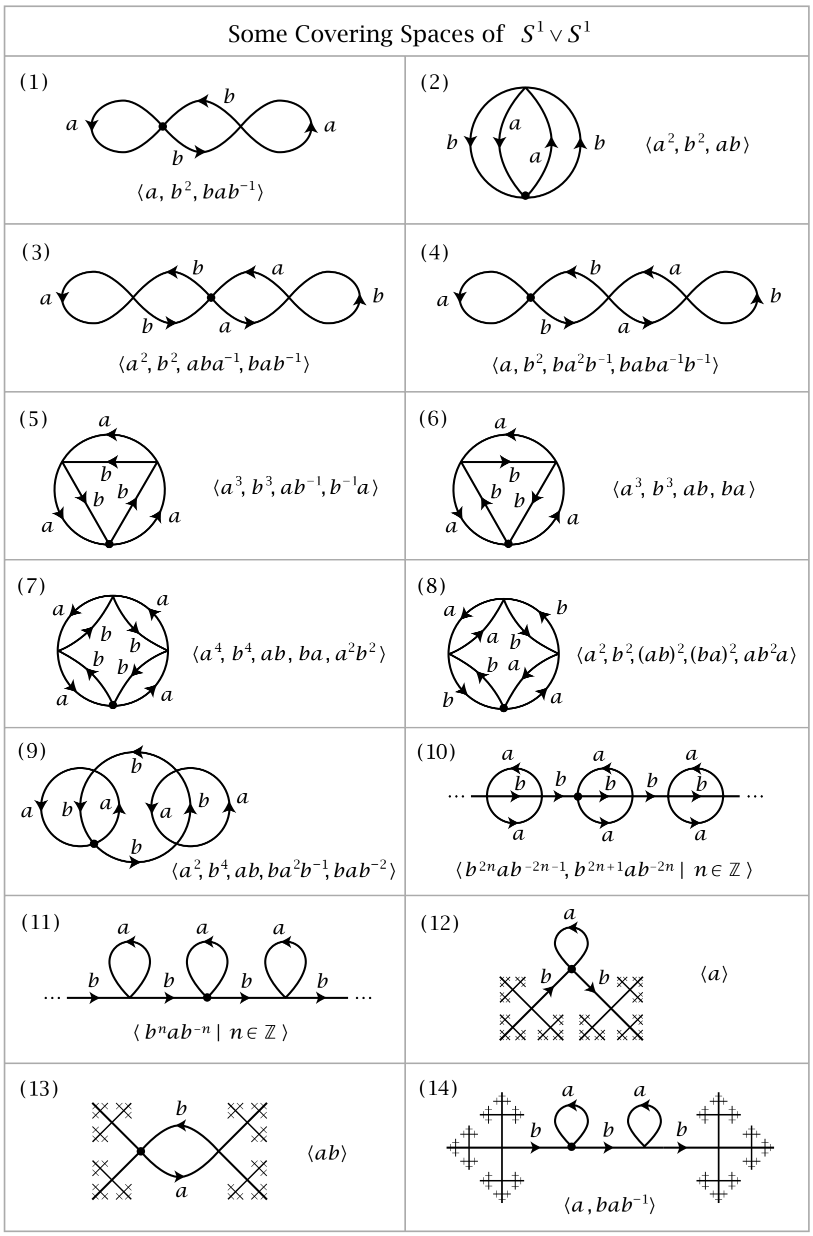
(c) Show that $L_{\overline{\alpha \cdot \beta}} = L_{\overline{\alpha}} \circ L_{\overline{\beta}}$.
(Note that we had to replace $\alpha$ by its inverse $\overline{\alpha}$ to make this relationship covariant).

(d) Now let us restrict to classes $[\gamma] \in \pi_1(X, x_0)$. Conclude that the assignment

$$\begin{aligned} \pi_1(X, x_0) &\longrightarrow \{\text{Permutations of } p^{-1}(x_0)\} \\ [\gamma] &\longmapsto L_{\overline{\gamma}} \end{aligned}$$

defines a group action of $\pi_1(X, x_0)$ on the set $p^{-1}(x_0)$.

(e) Choose five covers $\tilde{X}$ of $S^1 \vee S^1$ from Hatcher's table from p58 (copied below). Describe the permutation on the vertices of $\tilde{X}$ defined by the generator $a$, and the permutation defined by the generator $b$. No justification necessary; just state your answer.



(f) Recall the map $\Phi$ defined in Homework 5 Assignment Problem 2(d)

$$\begin{aligned} \Phi : \pi_1(X, x_0) \text{ mod } H &\longrightarrow p^{-1}(x_0) \\ H[\gamma] &\longmapsto \tilde{\gamma}(1) \end{aligned}$$

that defined a bijection between $p^{-1}(x_0)$ and the right cosets of $H = p_*(\pi_1(\tilde{X}, \tilde{x_0})$. Show that the group action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ defined above corresponds to the usual action of $\pi_1(X, x_0)$ on the right cosets by right multiplication,

$$[\gamma] \cdot (H[\beta]) = H[\beta \cdot \overline{\gamma}]$$

(g) Deduce that, if $H$ is normal in $\pi_1(X, x_0)$, the action of $\pi_1(X, x_0)$ induces a well-defined action by the quotient group $\pi_1(X, x_0)/H$.

In fact, we can reconstruct the cover $p : \tilde{X} \to X$ from the action of $\pi_1(X, x_0)$ on the fibre $F = p^{-1}(x_0)$ by taking a suitable quotient of $\tilde{X_0} \times F$, where $\tilde{X_0}$ is the universal cover. (This construction is described on p69-70 of Hatcher). Hatcher concludes that the $n$-sheeted covers of $X$ are classifed by conjugacy classes of group homomorphisms from $\pi_1(X, x_0)$ to the symmetric grrroup $S_n$.

3. (a) Let $X$ be a wedge of $n$ circles, so $\pi_1(X, x_0) = F_n$. Let $h : F_n \to G$ be a surjective group homomorphism. Explain how we could use the results of Assignment Problems 2 and Homework #6 Problem 3 to construct the graph $\tilde{X}$ covering $X$ with fundamental group the subgroup ker$(h) \subseteq \pi_1(X, x_0)$. Explain moreover how we can use the cover $\tilde{X}$ to determine a free generating set for ker$(h)$.

   (b) **(Topology QR Exam, May 2017).** Let $F$ be the free group on $a, b$. Let $G = \{1, x, x^2\}$ be the cyclic group on three generators written multiplicatively. Let $h : F \to G$ be a homomorphism which sends $a \mapsto x, b \mapsto x^2$. Find free generators of Ker$(h)$.

   (c) **(Topology QR Exam, Jan 2017).** Let $F$ be the free group on $a, b$. Let $G$ be a symmetric group (=group of all permutations) on three elements, and let $x, y \in G$ be elements of order 2 and 3, respectively. Let $h : F \to G$ be a homomorphism which sends $a \mapsto x, b \mapsto y$. Find free generators of Ker$(h)$.

4. (a) Let $F_n$ be the free group on $n$ generators, and $F_m$ the free group on $m$ generators. Show that, if $F_m \cong F_n$, the $m = n$. Conclude that the number $n$ (called the *rank* of $F_n$ ) is an isomorphism invariant. *Hint:* abelianization & structure theorem for finitely generated abelian groups

   (b) Let $X$ be a connected, finite graph with $n$ vertices and $e$ edges. Show that $\pi_1(X)$ is the free group of rank $(e - n + 1)$. You may assume the following result from graph theory.

   **Proposition (Combinatorics of trees).** Let $T$ be a finite tree (which is by definition connected). If $T$ has $n$ vertices, then it has $(n - 1)$ edges.

   (c) Let $X$ be a connected $m$-sheeted cover of the wedge $\bigvee_n S^1$ of $n$ circles. We proved on Homework 6 that $\pi_1(X)$ is a free group. What is the rank of $\pi_1(X)$?

   (d) Prove the following theorem.

   **Theorem (Schreier index formula).** Let $F_n$ be the free group of rank $n$. A subgroup of index $m \in \mathbb{N}$ in $F_n$ has rank $1 + m(n - 1)$. An infinite-rank subgroup has infinite index.

   This theorem shows that the rank of a finite-index subgroup is a function of its index. Moreover, the larger the index (so "smaller" the subgroup in $F_n$), the larger its rank!

   (e) Show by example that an infinte-index subgroup of $F_n$ can have finite or infinite rank. *Hint:* See Hatcher's table of covering spaces.

   (f) **(QR Exam, Aug 2021).** Let $F_n$ denote the free group on $n$ letters $\{a, b, c, \ldots\}$.

      (i) Prove that $F_4$ does **not** have a finite-index subgroup isomorphic to $F_8$.

(ii) Construct a finite-index subgroup $H$ of $F_4$ isomorphic to $F_7$. Determine (explaining your steps) a free generating set for $H$, and explain whether $H$ is normal.

5. **(Covering spaces as quotients by covering actions).** You may refer to Hatcher p71-73 while you write your solution. See Warm-Up Problem 7 for equivalent definitions of a group action on a space.

(a) **Definition (Orbit space).** Let $G$ be a group acting on a space $Y$. Recall that the *orbit* of a point $y \in Y$ is the subset

$$G \cdot y = \{g \cdot y \mid g \in G\} \subseteq Y.$$

The *orbit space* of this action, denoted $Y/G$, is the quotient space of $Y$ in which every orbit is collapsed to a point.

Let $p : \tilde{X} \to X$ be a surjective normal covering map with deck group $G(\tilde{X})$. Verify that we can identify $X$ with the orbit space $\tilde{X}/G(\tilde{X})$, and $p : \tilde{X} \to X$ with the quotient map.
*Hint:* Warm-up problem 9.

(b) **Definition (Covering space action).** Let $G$ be a group acting on a space $Y$. Then this action is *covering space action* if it satisfies the following condition. Each $y \in Y$ has a neighbourhood $U$ such that all images $g(U)$ for distinct $g \in G$ are disjoint. In other words, $g_1(U) \cap g_2(U) \neq \varnothing$ implies $g_1 = g_2$.

Let $p : \tilde{X} \to X$ be a connected covering space. Verify that the action of the deck group $G(\tilde{X})$ is a covering space action.

(c) Now suppose that a group $G$ is acting on a space $Y$ by a covering space action. Prove that the quotient map $p : Y \to Y/G$ is a normal covering space.

(d) Let $G$ is acting on a space $Y$ by a covering space action, and suppose $Y$ is path-connected. Prove that the Deck group of the cover $p : Y \to Y/G$ is isomorphic to $G$.

(e) Let $G$ is acting on a space $Y$ by a covering space action, and suppose $Y$ is path-connected and locally path-connected. Let $p : Y \to Y/G$ be the quotient. Prove that

$$\frac{\pi_1(Y/G, G \cdot y_0)}{p_*(\pi_1(Y, y_0))} \cong G.$$

In particular, if $Y$ is simply-connected, then $\pi_1(Y/G) \cong G$. *Hint:* This part is a result from class.

6. **(Bonus).**

(a) Let $X$ and $Y$ be path-connected topological spaces. Suppose that $f : X \to Y$ is a homotopy equivalence. Note that $f$, its homotopy inverse, and the associated homotopies need not respect any basepoints. Consider the induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ for $x_0 \in X$. To what extent do these groups, and the induced map, depend on choice of basepoint $x_0$? To what extent does the induced map on conjugacy classes of subgroups of the fundamental groups depend on choice of basepoint? Discuss. Give precise statements and justify them.

(b) Let $X$ and $Y$ be path-connected, locally path-connected, semi-locally simply connected topological spaces. Suppose that $f : X \to Y$ is a homotopy equivalence. Let $H$ be a subgroup of $X$ and $H'$ be its image under the induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ for some basepoint $x_0 \in X$. Are the covering spaces of $X$ and $Y$ associated to the conjugacy classes of $H$ and $H'$ (respectively) necessarily homotopy equivalent?