Terms and concepts covered: n-simplex; vertices, subsimplices, and faces; boundary and interior of a simplex; Δ-complex; chain complex, n-chains, exactness, homology groups; boundary homomorphisms, cycles and boundaries, simplicial homology groups, homology classes.

Corresponding reading: Hatcher Chapter 2, Introduction and Section 2.1, "Δ-Complexes" and "Simplicial Homology".

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. **Definition (Convex).** A subset C in Euclidean space is *convex* if it contains the line segment connecting any pair of its points.

   Definition (Convex combination, convex hull). Let X be a subset of Euclidean space. A *convex combination* of points in X is a sum of the form

   $$t_1x_1 + t_2x_2 + \cdots t_nx_n \qquad \text{such that } x_i \in X, t_i \in \mathbb{R}, t_i \geq 0, t_1 + \cdots + t_n = 1.$$

   The *convex hull* of X is the set of all convex combinations of points in X.

   (a) Prove that the convex hull of X is the minimal (under inclusion) convex subset containing X.

   (b) Prove that the convex hull of X is the intersection of all convex subsets containing X.

2. Let $\Delta^n = [v_0, v_1, \dots, v_n]$ be the standard n-simplex,

   $$\Delta^n = \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; t_i \geq 0, \sum_i t_i = 1 \right\}.$$

   What are its vertices? Show that the convex hull of any $(k+1)$ of its vertices is canonically homeomorphic to a k-simplex. Conclude that it therefore makes sense topologically (as well as combinatorially) to call this subspace a k-dimensional subsimplex.

3. Let $\Delta^n = [v_0, v_1, \dots, v_n]$ be an n-dimensional simplex. For each $k \leq n$, how many k-dimensional subsimplices does Δ^n have?

4. Describe the canonical Δ-complex structure on an n-simplex. What is its k-skeleton?

5. In this question, we will find another way to coordinatize an n-simplex. Let

   $$\Delta^0_* = \{0\},$$

   $$\Delta^n_* = \{ (s_1, \dots, s_n) \in \mathbb{R}^n \mid 0 \leq s_1 \leq s_2 \leq \dots \leq s_n \leq 1 \}$$

   (a) Draw Δ^n_* for $n = 0, 1, 2, 3$.

   (b) Recall that we defined the standard simplex $\Delta^n = \{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_i t_i = 1 \}$. Show that Δ^n is homeomorphic to Δ^n_* via the map

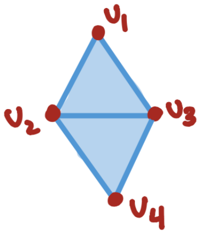
   $$s_i = t_0 + t_1 + \cdots + t_{i-1}.$$

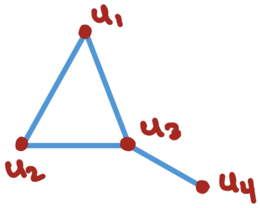
6. Consider our coordinatization of the n-simplex,

   $$\Delta^n_* = \{ (s_1, \dots, s_n) \in \mathbb{R}^n \mid 0 \leq s_1 \leq s_2 \leq \dots \leq s_n \leq 1 \}.$$

   Prove that the boundary $\partial\Delta^n$ and the open simplex $\mathring{\Delta}^n$ are indeed the boundary and interior of Δ^n, respectively, in the usual sense of point-set topology, when Δ^n is viewed as the subset Δ^n_* of $\mathbb{R}^n$.

7. (a) Verify that an n-simplex (as a topological space) is homeomorphic to a closed n-ball.
   (b) Verify that a Δ-complex structure on a space X is, in particular, a CW complex structure.
8. (a) Which of our standard CW complex structures on the spheres S^1 and S^2 are Δ-complex structures?
   (b) Is our standard CW complex structure on $\mathbb{R}\mathrm{P}^2$ a Δ-complex structure?
9. Choose your preferred name for the ∂ symbol.
10. Let X be either of the Δ-complexes shown below. Let $C_n(X)$ denote the associated nth simplicial chain group, and let $\partial_n : C_n(X) \to C_{n-1}(X)$ be the boundary map.





   (a) We choose the total orderings of the vertices v_1, v_2, v_3, v_4 and u_1, u_2, u_3, u_4, respectively, for the two complexes. Explain how this determines an order on the vertices of every simplex. Label the edges of each complex with the appropriate direction.
   (b) Compute the boundary (that is, the image under ∂_n) of the following n-chains.
      (i) $2[v_1, v_2, v_3]$
      (ii) $[v_1, v_2, v_3] + [v_2, v_3, v_4]$
      (iii) $[v_1, v_2, v_3] - [v_2, v_3, v_4]$
      (iv) $[u_1, u_2] - [u_1, u_3] + [u_2, u_3]$
      (v) $[u_1, u_2] - [u_1, u_3] + [u_2, u_3] + [u_3, u_4]$
   (c) Explain for each calculation how this boundary relates to your intuitive geometric understanding of "boundary".
11. Let X be a Δ-complex. Let $C_n(X)$ denote the nth simplicial chain group, and let $\partial_n : C_n(X) \to C_{n-1}(X)$ be the boundary map.
   (a) Verify that $\partial_n \circ \partial_{n+1} = 0$.
   (b) Give a geometric interpretation of the equation $\partial_n \circ \partial_{n+1} = 0$, in the spirit of "a boundary has no boundary".
12. (a) Let u, v be vertices in a simplicial complex joined by an edge. What is the relationship between the (oriented) edge $[u, v]$ and the (oriented) edge $[v, u]$? What is the relationship between the 1-chains $[u, v]$ and $-[v, u]$, and the relationships between their boundaries?
   (b) Let $\Delta_n = [v_0, v_1, \ldots, v_n]$ be an n-simplex, and τ a permutation in S_{n+1}. Show that

$$\partial_n([v_{\tau(0)}, v_{\tau(1)}, \ldots, v_{\tau(n)}]) = \begin{cases} \partial_n([v_0, v_1, \ldots, v_n]) & \text{if } \tau \text{ is an even permutation} \\ -\partial_n([v_0, v_1, \ldots, v_n]) & \text{if } \tau \text{ is an odd permutation.} \end{cases}$$

   Conclude that our ordering of the vertices does matter in our computation of the differential—different orders result in different signs—but order does not matter up to even permutations.
13. Let (C_*, d_*) be a chain complex, and suppose it is exact at every point C_n. Such sequences are called *exact sequences*. What is the homology of (C_*, d_*)?

14. Let (C_*, d_*) be a chain complex supported in degree n, that is,

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow C_n \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0.$$

    What is the homology of (C_*, d_*)?

15. Let (C_*, d_*) be a chain complex.
    (a) Suppose that the differential d_n is identically zero for some n.

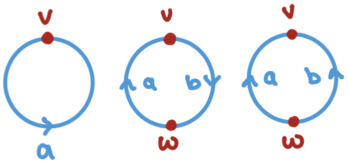
$$\cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n=0} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

    Show that $H_{n-1} = \ker(d_{n-1})$, and $H_n = C_n/\mathrm{im}(d_{n+1})$.
    (b) Suppose the differential d_n is identically zero for every n. Show that $H_n = C_n$ for every n.

16. Compute the simplicial homology of the disjoint union of n points.

17. Compute the simplicial homology of a 1-simplex.

18. Compute the simplicial homology of S^1 with each of the following Δ-complex structures, with the given orientations of the edges.



19. Compute the simplicial homology groups of the wedge $\bigvee_k S^1$ of k circles.

20. Let X be a Δ-complex, and $C_n(X)$ its n^{th} simplicial chain group.
    (a) Show that $C_0(X) = \ker(\partial_0)$, so $C_0(X)$ is the group of 0-cycles. Conclude that, topologically, a 0-cycle is any linear combination of vertices of X.
    (b) Show that two vertices in $C_0(X)$ are homologous exactly if they are connected via a path of edges in X.
    (c) Conclude that $H_0(X)$ consists of formal sums of equivalence classes of vertices of X, where two vertices are equivalent if they are in the same path-component of X.
    (d) Explain the sense in which $H_0(X)$ "is" the free abelian group on the path components of X.

21. (a) Let (C_*, d_*) be a chain complex. Explain why, if the n^{th} homology group H_n has rank N, then the n^{th} chain group C_n must have had rank at least N.
    (b) Let X be a space. We will show that the homology groups are homeomorphism invariants (in fact, homotopy invariants). Explain why, if if the n^{th} simplicial homology group $H_n(X)$ has rank N, then any Δ-complex structure on X must have at least N simplices of dimension n.

22. Review the structure theorem for finitely generated abelian groups.

23. Let A be a square matrix with entries in a commutative unital ring R. Recall that A is *invertible over* R if it has a 2-sided inverse matrix with entries in R.
    (a) Suppose that A has entries in $\mathbb{Z}$, so we may view A as a matrix over $\mathbb{Z}$ or over $\mathbb{Q}$. Show by example that A may be invertible over $\mathbb{Q}$ but not over $\mathbb{Z}$. Explain why, if A is invertible over $\mathbb{Z}$, it is necessarily invertible over $\mathbb{Q}$.
    (b) Show that A is invertible over R if and only if its determinant is a unit in R. In particular, a matrix with entries in $\mathbb{Z}$ is invertible over $\mathbb{Z}$ if and only if it has determinant ± 1.
    (c) Explain why a matrix with entries in a field k is invertible over k if and only if it is invertible over any field extension of k.

Assignment questions

(Hand these questions in!)

0. **(Optional).** Submit course feedback to Jenny in our (optional, anonymous) midterm evaluation survey:

   `https://forms.gle/6SsfEGps6rULWW9t5`

1. **(Covering spaces as quotients by covering actions, ctd).** This is a continuation of Assignment Problem 5 from Homework #7. You may refer to Hatcher p72-73 while you write your solution to this problem.
   (a) On Homework #5 Problem 2(f), you constructed the covers of the torus associated to the subgroups $4\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times 4\mathbb{Z}$, and $2\mathbb{Z} \times 2\mathbb{Z}$ of its fundamental group $\mathbb{Z}^2$. Briefly explain/illustrate how you could construct these covering spaces using a suitable action of the groups $4\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times 4\mathbb{Z}$, and $2\mathbb{Z} \times 2\mathbb{Z}$, respectively, on the universal cover $\mathbb{R}^2$ of the torus.
   (b) Suppose we have a covering space action of a group G on a simply connected space Y. Let $H_1 \subseteq H_2 \subseteq G$ be subgroups. Briefly explain how to use the action of H_1, H_2 on Y to construct the intermediate cover $q : X_1 \to X_2$ defined in Homework 7, Assignment Problem #1. What happens in the special cases $H_1 = 0$ or $H_2 = G$? You do not need to check details.
   (c) Prove the following result.

   Proposition. A free action of a finite group on a Hausdorff space Y is a covering space action.

   (d) In 1-2 sentences, explain why, for $n \geq 2$, the defining quotient map $S^n \to \mathbb{RP}^n$ is the universal cover.
   Remark: In contrast, the map $S^{2n+1} \to \mathbb{CP}^n$ is not a covering map. Its fibres are not discrete.

2. For each of the following spaces, define a generalized simplicial complex structure on the space, and compute its simplicial homology groups.

   (a) a 2-simplex (b) S^2 (c) a torus (d) a Mobius band

3. **Definition (Morphism of chain complexes).** A *morphism f_* of chain complexes* or *chain map* from (C_*, ∂_*) to (D_*, δ_*) is a sequence of group homomorphisms $f_n : C_n \to D_n$ making the following diagram commute.

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r] & D_{n+1} \arrow[r, "\delta_{n+1}"] & D_n \arrow[r, "\delta_n"] & D_{n-1} \arrow[r, "\delta_{n-1}"] & \dots
\end{tikzcd}$$

   (a) Verify that a morphism f_* of chain complexes induces, for each n, well-defined group homomorphisms on the homology groups

$$f_n : H_n(C_*) \to H_n(D_*).$$

   (b) **Definition (Quasi-isomorphism).** A morphism of chain complexes $f_* : (C_*, \partial_*) \to (D_*, \delta_*)$ is a *quasi-isomorphism* if the maps induced on homology are all isomorphisms.

   Give an example of a quasi-isomorphism of chain complexes where at least one map f_n is not an isomorphism.

4. **(Homomorphisms of free abelian groups).** Let A be an $n \times n$ integer matrix, viewed as $\mathbb{Z}$-linear map from $\mathbb{Z}^n$ to $\mathbb{Z}^n$.

(a) Suppose that A has rank n. Prove that the kernel of A is trivial.
(Note: Here we mean 'rank' in the usual sense from linear algebra, for example, it is the rank of A when A is viewed as a matrix with entries in $\mathbb{Q}$).

(b) Show by example that, even if A has rank n, it need not be surjective.

(c) The *cokernel* of a map of abelian groups is the quotient of its codomain by its image. Prove or find a counterexample: if the map A has rank n, then the cokernel of A must be finite.

(d) Prove the $\mathbb{Z}$-module version of the rank-nullity theorem: If A is an $(m \times n)$ matrix of rank k, then its image is a free abelian subgroup of $\mathbb{Z}^m$ of rank k, and its kernel is a free abelian subgroup of $\mathbb{Z}^n$ of rank $(n - k)$.

5. Let A be a square matrix with entries in a commutative unital ring R. Recall we say A is *invertible over R* if it has a 2-sided inverse matrix with entries in R. See Warm-up Problem 23.

Definition / Theorem (Smith normal form). Let A be an $m \times n$ matrix over a principal ideal domain R. There exists an $m \times m$ matrix S and an $n \times n$ matrix T such that S and T are invertible over R, and

$$SAT = \begin{bmatrix} \alpha_1 & 0 & 0 & & \cdots & & 0 \\ 0 & \alpha_2 & 0 & & \cdots & & 0 \\ 0 & 0 & \ddots & & & & 0 \\ \vdots & & & \alpha_r & & & \vdots \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & & \cdots & & 0 \end{bmatrix}$$

where the diagonal entries α_i satisfy $\alpha_i | \alpha_{i+1}$ for all $1 \leq i \leq r$. The matrix A is called the *Smith normal form* of A. The elements α_i are unique up to multiplication by a unit in R. They are called the *invariant factors* of A.

We are interested in the case $R = \mathbb{Z}$.
Note that, since S, T are invertible, the rank of A is equal to the rank of its Smith normal form.

(a) Let A be a $\mathbb{Z}$-linear map $\mathbb{Z}^n \to \mathbb{Z}^m$ with invariant factors $\alpha_1, \alpha_2, \ldots \alpha_r$. Prove that the cokernel of A is isomorphic to $\mathbb{Z}^{m-r} \oplus \bigoplus_i \mathbb{Z}/\alpha_i\mathbb{Z}$. Conclude that Smith normal form can therefore be used to put a quotient of a free abelian group $\mathbb{Z}^m$ into standard form (standard in the sense of the structure theorem for finitely generated abelian groups), by writing generators for the kernel as the columns of a matrix.

Remark: In fact, any proof of the structure theorem is likely implicitly a proof of existence/uniqueness of Smith normal form.

(b) An integer matrix can be put in Smith normal form using the following row and column operations, which are invertible over $\mathbb{Z}$.

R1. swap rows R_i and row R_j
R2. multiply row R_i by -1
R3. replace row R_i by $R_i + nR_j$ for some row $R_j \neq R_i$ and $n \in \mathbb{Z}$

C1. swap columns C_i and row C_j
C2. multiply column C_i by -1
C3. replace column C_i by $C_i + nC_j$ for some row $C_j \neq C_i$ and $n \in \mathbb{Z}$

To transform A into its Smith normal form, we use the following general steps. You may (if you wish) read a detailed description in the following handout

`https://www3.nd.edu/~sevens/smithform.pdf`

.

- Let d be the gcd of all entries of A. Use row and column operations, and the Euclidean algorithm, to transform the matrix so that some matrix entry equal to d.

Remark: Observe that the row and column operations do not change the gcd.

- Use row and column swaps (R1 and C1) to place d in entry $(1, 1)$.
- Use row and column operations R3 and C3 to clear the first row and first column, to obtain a matrix of the form
$$\begin{bmatrix} d & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{bmatrix}.$$
- Repeat the procedure on the matrix A'.

  Remark: Each row operation corresponds to multiplying A on the left by an invertible integer *elementary matrix*. Each column operation corresponds to multiplying A on the right by an invertible integer *elementary matrix*. Thus, by keeping track of the sequence of row and column operations applied, we can determine the matrices S and T as products of elementary matrices.

Explain and illustrate the steps to transform the following matrix into its Smith normal form.

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 8 & 4 & 12 \end{bmatrix}$$

(You do not need to compute S and T). Verify your answer by going to the website

`https://sagecell.sagemath.org/`

and entering the lines

```
A = matrix([[4, 6, 6],[8, 4, 12]])
A.smith_form()
```

When you hit "Evaluate", SAGE will give you three matrices: the Smith normal form of A, and the matrices T and S.

(c) Let A be an $m \times n$ integer matrix, and let B be an $\ell \times m$ integer matrix, such that $BA = 0$.

$$\begin{tikzcd} \mathbb{Z}^n \arrow[r, "A"] \arrow[rr, bend right, "0"'] & \mathbb{Z}^m \arrow[r, "B"] & \mathbb{Z}^\ell \end{tikzcd}$$

Prove that B factors through a $\mathbb{Z}$-linear map $\overline{B} : \mathbb{Z}^m/\mathrm{im}(A) \to \mathbb{Z}^\ell$, and that

$$\ker(\overline{B}) = \ker(B)/\mathrm{im}(A).$$

(d) Prove the following.

Theorem (Smith normal form and homology computations). Let A be an $m \times n$ integer matrix, and let B be an $\ell \times m$ integer matrix, such that $BA = 0$.

$$\begin{tikzcd} \mathbb{Z}^n \arrow[r, "A"] \arrow[rr, bend right, "0"'] & \mathbb{Z}^m \arrow[r, "B"] & \mathbb{Z}^\ell \end{tikzcd}$$

Then

$$\ker(B)/\mathrm{im}(A) = \mathbb{Z}^{m-r-s} \oplus \bigoplus_{i=1}^{r} \mathbb{Z}/\alpha_i\mathbb{Z}$$

where $r = \mathrm{rank}(A)$, $s = \mathrm{rank}(B)$, and $\alpha_1, \ldots, \alpha_r$ are the invariant factors of A.

(e) Use part (d) and SAGE to compute the homology of the following chain complex.

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} -30 & -54 \\ -16 & -55 \\ 3 & 9 \\ -2 & 7 \end{bmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{bmatrix} 41 & -90 & -178 & -162 \\ 34 & -74 & -144 & -134 \end{bmatrix}} \mathbb{Z}^2 \longrightarrow 0$$

6. **(Bonus).** Let K be a compact space with $k_0 \in K$. Let X be a CW complex. Let $f : K \to X$ be a continuous map.
   (a) Suppose K is path-connected and locally path-connected. Prove that the image $f_*(\pi_1(K, k_0))$ is finitely generated.
   (b) Now only assume that K is path-connected. Must the group $f_*(\pi_1(K, k_0))$ be finitely generated?