Terms and concepts covered: singular n-chains, singular homology groups. Induced maps on homology. Chain homotopy. Reduced homology groups. Good pair, long exact sequence of a pair.

Corresponding reading: Hatcher Ch 2.1, "Singular homology", "Homotopy invariance", "Exact sequences and excision" to end of page 114, Ch 2.A "Homology and fundamental group".

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Show that, if a 2-simplex T in a Δ-complex is glued along the word in the edges $a_1a_2a_3$, then $\partial T = a_1 + a_2 + a_3$.
2. (a) Given a chain complex,

$$\dots \xrightarrow{d_{n+2}} C_{n+1}(X) \xrightarrow{d_{n+1}} C_n(X) \xrightarrow{d_n} C_{n-1}(X) \xrightarrow{d_{n-1}} \dots$$

   explain why the homology group $H_n(X)$ depends only on the groups $C_{n+1}(X)$, $C_n(X)$, $C_{n-1}(X)$, and the maps d_{n+1} and d_n.

   (b) Let X be a generalized simplicial complex. We proved that a generating set for $\pi_1(X)$ is determined by its 1-skeleton X^1, and that the relations for $\pi_1(X)$ (and hence the isomorphism type) are determined by the 2-skeleton X^2.
   Let $H_n(X)$ be the nth simplicial homology group of X. Explain the sense in which generators for $H_n(X)$ (cycles) are determined by the n-skeleton X^n, and relations for $H_n(X)$ (boundaries) are determined by the $(n+1)$-skeleton X^{n+1}.
3. Let X be a space with a choice of generalized simplicial complex structure. Explain the difference between the definitions of the simplicial n-chains on X, and the singular n-chains on X.
4. Let X be a space. Let $C_n(X)$ denote the singular n-chains on X, and let $H_n(X)$ denote the nth singular homology group. Suppose that X has path components $\{X_\alpha\}$.
   (a) Why must the image of each singular n-chain be contained in a single path-component X_α?
   (b) Fix n. Deduce that, as a group, $C_n(X)$ decomposes as a direct sum $C_n(X) = \bigoplus_\alpha C_n(X_\alpha)$.
   (c) Verify that the boundary map ∂_n respects this decomposition.
   (d) Conclude that there is a decomposition $H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$
5. Let X be a point. Working directly from the definition of singular homology, show that

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \geq 1. \end{cases}$$

6. (a) Let X be a path-connected space, and let $H_n(X)$ denote its nth singular homology group. Working directly from the definition of singular homology, show that $H_0(X) \cong \mathbb{Z}$.
   (b) Let X be a space with path-components $\{X_\alpha\}_\alpha$. Use part (a) and Warm-up Problem 4 to show that $H_0(X) \cong \bigoplus_\alpha \mathbb{Z}$.
7. Let $f : X \to Y$ be a continuous map of topological spaces, and let $f_\#$ denote the map induced by f on singular n-chains,

$$\begin{aligned} f_\# : C_n(X) &\longrightarrow C_n(Y) \\ [\sigma : \Delta^n \to X] &\longmapsto [f \circ \sigma : \Delta^n \to Y]. \end{aligned}$$

(a) Verify that $f_\# \circ \partial = \partial \circ f_\#$.

(b) Conclude that $f_\#$ is a chain map, so for each n, there is an induced group homomorphism $f_* : H_n(X) \to H_n(Y)$.

8. Fix n. For a continuous map $f : X \to Y$ of topological spaces, let $f_* : H_n(X) \to H_n(Y)$ denote the induced map on singular homology groups, as in Warm-up Problem 7.

   (a) For maps of spaces $g : X \to Y$ and $f : Y \to Z$, verify that $(f \circ g)_* = f_* \circ g_*$.

   (b) Verify that $id_X : X \to X$ induces the identity map on $H_n(X)$.

   (c) Conclude that H_n is a functor from the category of topological spaces and continuous maps, to the category of abelian groups and group homomorphisms.

9. Let $f : X \to Y$ be a continuous map of path-connected spaces. Show that the induced map $f_* : H_0(X) \to H_0(Y)$ is an isomorphism.

10. Let $\iota : A \hookrightarrow X$ be the inclusion of a subspace A of a space X. Show that, if A is a retract of X, then the induced map $\iota_* : H_n(A) \to H_n(X)$ is injective for all n.

11. We sketched a proof in class of the following result.

    Theorem (homotopic maps induce the same map on H_n). If $f, g : X \to Y$ are homotopic maps, then they induce the same map $f_* = g_*$ on singular homology groups.

    Show that this theorem (and functoriality of H_n) implies the following.

    Theorem (H_n is a homotopy invariant). Let $f : X \to Y$ be a homotopy equivalence. Then the induced map on singular homology $f_* : H_n(X) \to H_n(Y)$ is an isomorphism. In particular, homotopy equivalent spaces have isomorphic homology groups.

12. Let $f : X \to Y$ be a nullhomotopic map. Show that the induced map $f_* : H_n(X) \to H_n(Y)$ is zero for all $n \geq 1$, and that the induced map $f_* : \widetilde{H}_0(X) \to \widetilde{H}_0(Y)$ is zero. What is the induced map $f_* : H_0(X) \to H_0(Y)$?

13. **(Interpreting exact sequences).** Prove that ...

    (a) the sequence
    $$0 \longrightarrow A \xrightarrow{f} B$$
    is exact if and only if f is injective.

    (b) the sequence
    $$B \xrightarrow{g} C \longrightarrow 0$$
    is exact if and only if g is surjective.

    (c) the sequence
    $$0 \longrightarrow A \xrightarrow{h} B \longrightarrow 0$$
    is exact if and only if h is an isomorphism.

    (d) the sequence
    $$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$
    is exact if and only if f is injective, g is surjective, and $C \cong B/f(A)$, where $f(A) \cong A$.

14. **(Calculations with exact sequences of abelian groups).** The following sequences are exact.

    (a) Compute the group A. *Hint:* Which maps must be injective, surjective, zero?
    $$\ldots \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/5\mathbb{Z} \longrightarrow A \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \ldots$$

(b) Compute the group B.

$$\dots 0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow B \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/5\mathbb{Z} \longrightarrow 0 \longrightarrow \dots$$

(c) What are the possibilities for the group C?

$$\dots \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow C \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow \dots$$

15. Consider the maps $m : S^1 \to T$ and $\ell : S^1 \to T$, the inclusions of the meridian $S^1 \times \{1\}$ and longitudinal circle $\{1\} \times S^1$, respectively. See Assignment Problem 3. Explain how the induced maps $H_1(S^1) \to H_1(T)$ give a topological interpretation for the homology classes in $H_1(T)$. In general, we can sometimes understand degree-n homology classes in X in terms of the induced maps from a closed n-manifold.

16. Once you Assignment Problem 1:

(a) Convince yourself that the steps in your argument would carry over directly to simplicial homology.

(b) Explain how the construction and the conclusions would generalize to a general chain complex of abelian groups (C_*, d_*) such that $C_n = 0$ for $n < 0$ and C_0 is free abelian with a distinguished basis.

Assignment questions

(Hand these questions in!)

1. **Definition (Reduced homology).** Let X be a space, and let $C_n(X)$ denote its nth singular chain group. Define the *augmented* singular chain complex

$$\cdots \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where $\epsilon(\sum_i n_i \sigma_i) = \sum_i n_i$.

The *reduced* singular homology groups $\widetilde{H}_n(X)$ of X are the homology groups of this chain complex.

(a) Let C be a free abelian group with $\mathbb{Z}$-basis B, and let $\epsilon : C \to \mathbb{Z}$ be the homomorphism mapping every basis element to 1. The kernel K of ϵ is called the *augmentation ideal*. Show that K is generated by the elements $a - b$ for $a, b \in B$, and show that, given a distinguished element $b_0 \in B$, the set $\{b - b_0 \mid b \in B, b \neq b_0\}$ is a $\mathbb{Z}$-basis for K.

(b) Verify that the augmented singular chain complex is, in fact, a chain complex.

(c) Suppose $X = \varnothing$. Verify that

$$H_n(\varnothing) = 0 \quad \text{for all } n, \quad \text{and} \quad \widetilde{H}_n(\varnothing) = \begin{cases} 0, n \neq -1 \\ \mathbb{Z}, n = -1 \end{cases}$$

(d) Suppose $X \neq \varnothing$. Prove that

$$H_n(X) = \widetilde{H}_n(X), \; n \neq 0$$
$$H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$$

In particular, the singular homology groups and reduced singular homology groups only differ mildly in degree zero! Nevertheless, the reduced homology have some favourable combinatorial properties that make them often more convenient to work with. One reason is the following.

(e) Let X be a contractible space. Show that $\widetilde{H}_n(X) = 0$ for all n.

Remark: The reduced homology groups $\widetilde{H}_n$ define functors from Top to Ab.

Remark: All the results of this exercise carry through if we replace "singular chains" (and "singular homology") by "simplicial chains" (and "simplicial homology") or—later in the class—"cellular chains" (and "cellular homology").

2. In this problem, we will begin a proof of the following theorem.

   Theorem ($H_1(X) \cong \pi_1(X, x_0)^{ab}$). Let X be path-connected space with basepoint x_0. There is a surjective group homomorphism

$$h : \pi_1(X, x_0) \longrightarrow H_1(X)$$
$$[\gamma] \longmapsto \text{singular 1-chain } \gamma$$

   whose kernel is the commutator subgroup of $\pi_1(X, x_0)$. In particular,

$$H_1(X) \cong \pi_1(X, x_0)^{ab}.$$

   You may read Hatcher 2.A and other relevant sections while you write your solutions.

   (a) Let α be a based loop in (X, x_0). Explain how α is a singular 1-chain, and verify that α is a cycle.

   (b) Suppose α is the constant loop at x_0. Show that α is a boundary, specifically, the boundary of the constant singular 2-simplex at x_0.

   (c) Show that, if $\alpha \simeq \beta$ are homotopic rel $\{0, 1\}$, then α and β differ by a boundary. *Hint:* Subdivide $I \times I$.

   (d) If α and β are paths with $\alpha(1) = \beta(0)$, then the 1-chain $\alpha \cdot \beta$ and the 1-chain $\alpha + \beta$ differ by a boundary. In particular, given based loops α, β, this result shows that the 1-cycle $\alpha \cdot \beta$ is homologous to the 1-cycle $\alpha + \beta$. *Hint:* Define a singular 2-simplex with boundary α, β, and $\alpha \cdot \beta$.

   (e) If α is a based path and $\overline{\alpha}$ its inverse, show that the 1-chain $\overline{\alpha}$ is homologous to $-\alpha$.

   (f) **(h is a homomorphism).** Deduce that h is a well-defined homomorphism. It is a special case of the *Hurewicz homomorphism*.

   (g) **(h is surjective).** Let $x \in H_1(X)$, and let $\sum_i n_i \sigma_i$ be a 1-cycle representing x. By allowing repeats of summands σ_i, we can assume each coefficient n_i is ± 1. Show that x is in the image of h. You may use these steps:
   - Explain why we may assume each n_i is 1.
   - Explain why we may assume each σ_i is a loop, by inductively replacing sums $\sigma_i + \sigma_j$ with the product of paths $\sigma_i \cdot \sigma_j$.
   - Explain why we can assume σ_i is a loop based at x_0, possibly by replacing σ_i by a homotopic loop of the form $\eta_i \cdot \sigma_i \cdot \eta_i^{-1}$.
   - Find $[\gamma] \in \pi_1(X, x_0)$ such that $h([\gamma]) = x$.

   (h) **($[\pi_1, \pi_1] \subseteq \ker(h)$).** Explain why the commutator subgroup of $\pi_1(X, x_0)$ must be contained in the kernel of h.

   Next week, we will complete this problem with a proof that $\ker(h) \subseteq [\pi_1, \pi_1]$.

3. (a) Let X be a generalized simplicial complex, and let $f : Y \to X$ be the inclusion of a subcomplex. Show that f induces a well-defined homomorphism on reduced simplicial homology groups,

$$f_* : \widetilde{H}_*(Y) \to \widetilde{H}_*(X).$$

   (b) Compute the maps induced on reduced homology by the following maps of topological spaces. *Hint:* For some of these maps, you can solve the problem by viewing their homology groups as abstract groups and considering the constraints on possible group homomorphisms. In other cases, use simplicial homology and your solution to part a.

   (i) The canonical quotient map $q : S^2 \to \mathbb{RP}^2$.
   (You can take for granted that this map also induces a well-defined map on homology!)

   (ii) The inclusion of the equator $f : S^1 \to S^2$.

   (iii) The map $m : S^1 \to T$, where $T = S^1 \times S^1$, and m is the inclusion of the meridian $S^1 \times \{1\}$.

4. (a) Prove the following proposition.

Proposition (Homology of a wedge sum). Let $\{X_\alpha\}$ be a collection of topological spaces, with basepoint $x_\alpha \in X_\alpha$ such that (X_α, x_α) is a good pair for each α. Let $\bigvee_\alpha X_\alpha$ be the wedge sum formed by identifying the basepoints x_α, and let $i_\alpha : X_\alpha \to \bigvee_\alpha X_\alpha$ be the inclusion map. Then for each n there is an isomorphism on homology

$$\oplus_\alpha (i_\alpha)_* : \bigoplus_\alpha \widetilde{H}_n(X_\alpha) \xrightarrow{\cong} \widetilde{H}_n\left(\bigvee_\alpha X_\alpha\right)$$

Hint: Consider the pair $(\bigsqcup_\alpha X_\alpha, \bigsqcup_\alpha \{x_\alpha\})$.

(b) State the homology groups of the following spaces. No justification needed.

(i) $\bigvee_k S^1$
(ii) a once-punctured torus
(iii) $S^1 \vee S^2 \vee S^\infty$
(iv) $S^1 \vee S^1 \vee S^2$
(v) the wedge sum of $\mathbb{R}\mathrm{P}^2$ and a Mobius band

(c) Verify that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in every degree. Show, however, that they are not homotopy equivalent. Deduce that, although homology groups are homotopy invariants, they are not *complete* invariants (i.e., they are not sufficient to distinguish all homotopy types).