Final Exam

Math 592
29 April 2025
Jenny Wilson

Name: ______________________________

Instructions: This exam has 6 questions for a total of 30 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting, and explicitly check that the hypotheses hold.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	3	
2	4	
3	3	
4	3	
5	7	
6	10	
Total:	30	

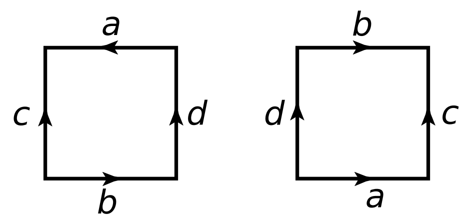
Notation

- $I = [0, 1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ (unit n-sphere) (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ complex projective n-space

1. (3 points) Let $T = S^1 \times S^1$ be the torus. Suppose that $f : T \to T$ is a continuous map that is **not** surjective. Show that the induced map $f_* : H_2(T) \to H_2(T)$ is zero.

2. (a) (1 point) State the classification theorem for closed surfaces.

(b) (3 points) A certain surface X is constructed by gluing together two squares by the edge identifications shown. Determine (with justification) which surface it is, in the sense of the classification theorem. You do not need to verify that X is a surface.



3. (3 points) Let $p : \widetilde{X} \to X$ be a (surjective) covering space map. Suppose that $\widetilde{X}$ is a finite tree, that is, a tree with finitely many vertices and edges. Prove that the covering space is trivial in the sense that p must be 1-sheeted.

4. (3 points) Let $p : E \to X$ be a covering space map of topological spaces that have the T_1 property (this means "points are closed"). Prove that p induces isomorphisms on local homology groups, that is, for all $e \in E$ and integers $q \geq 0$,

$$p_* : H_q(E, E \setminus \{e\}) \xrightarrow{\cong} H_q(X, X \setminus \{p(e)\}).$$

5. Fix $n \geq 1$ and consider complex projective space $\mathbb{C}\mathrm{P}^n$.

   (a) (2 points) Fix a point $x_0 \in \mathbb{C}\mathrm{P}^n$. Show that punctured complex projective space $\mathbb{C}\mathrm{P}^n \setminus \{x_0\}$ deformation retracts onto $\mathbb{C}\mathrm{P}^{n-1}$.

   (b) (5 points) Compute the homology of the connected sum $\mathbb{C}\mathrm{P}^n \# \mathbb{C}\mathrm{P}^n$.

   Recall that the connected sum of two m-dimensional (oriented) manifolds is obtained by deleting an open m-ball from each, and gluing together the boundaries of these balls via an (orientation-reversing) homeomorphism.

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Extra space for Problem 5.

6. (10 points) For each of the following statements: if the statement is true, write “`True`”. Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not true, you can receive partial credit for writing “`False`” without a counterexample.

   (a) Let $\mathcal{F}$ be a functor from the category of topological spaces and continuous maps, to the category of abelian groups and group homomorphisms. Then $\mathcal{F}$ must take the coproduct of spaces $X \coprod Y$ to the coproduct $\mathcal{F}(X) \coprod \mathcal{F}(Y)$ of abelian groups.

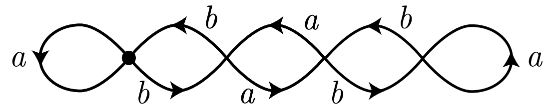
   (b) Let T denote the torus. For every $n \geq 1$, any continuous map $f : \mathbb{C}\mathrm{P}^n \to T$ is nullhomotopic.

   (c) There does not exist a CW complex with fundamental group $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$.

   (d) Suppose that a space X is the union of two open subsets U, V with nonempty intersection. If U and V are simply connected, then X must be simply connected.

   (e) Suppose that a certain based map $f : (X, x_0) \to (Y, f(x_0))$ of path-connected spaces X, Y induces the zero map on fundamental group. Then $f_* : H_1(X) \to H_1(Y)$ must be the zero map.

(f) Let $X = S^1 \vee S^1$ be the wedge of two circles, with wedge point x_0. We label the two oriented edges by a and b, respectively, and (by abuse of notation) also denote by a and b the corresponding elements of $\pi_1(X, x_0)$, so we can identify $\pi_1(X, x_0)$ with the free group $F_{\{a,b\}}$. Let H be the subgroup of $F_{\{a,b\}}$ associated with the based covering space $(\widetilde{X}, \widetilde{x_0})$ of X shown. Then every element of its normalizer $N(H)$ in $F_{\{a,b\}}$ is in fact contained in H.



(g) Let $(\widetilde{X}, \widetilde{x_0})$ and $H \subseteq F_{\{a,b\}}$ be as in part (f). Then every element of $F_{\{a,b\}}$ is contained in $N(H)$.

(h) Let $p : S^3 \to S^3$ be a map of degree $d > 1$. Then p cannot be a covering space map.

(i) Let X be a CW complex that has cells only in dimensions $0, 1, 2, 4$. Then $H_2(X)$ is a free abelian group (possibly zero).

(j) Let $A \subseteq X$ be topological spaces. If A is contractible, then $H_q(X, A) \cong \widetilde{H}_q(X/A)$ for all q.