Midterm Exam I

Math 592
12 February 2025
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Name: ______________________________

Instructions: This exam has 4 questions for a total of 20 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 90 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Question	Points	Score
1	3	
2	8	
3	2	
4	7	
Total:	20	

Notation

- $I = [0,1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ (unit n-sphere) (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{RP}^n$ real projective n-space
- $\mathbb{CP}^n$ complex projective n-space

1. (3 points) Let X be a space, and let $f : S^1 \to X$ be a continuous map. Prove that the following are equivalent. *Hint:* You can use without proof the fact that the quotient space $(S^1 \times I)/(S^1 \times \{1\})$ is a 2-disk.

   (i) View the domain of f as the boundary of the closed 2-disk D^2. The map f extends to a continuous map $D^2 \to X$.

   (ii) The induced map $f_* : \pi_1(S^1, s) \to \pi_1(X, f(s))$ is the zero map (with respect to any choice of basepoint $s \in S^1$).

Extra space for Problem 1.

2. For each of the following spaces X, compute the fundamental group.

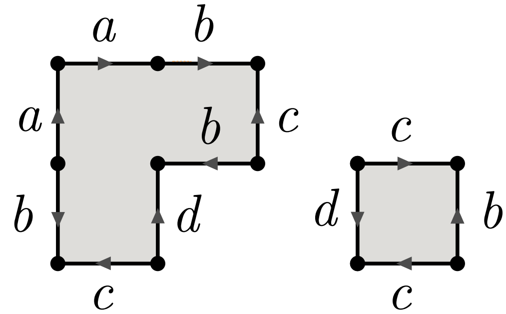
   You do not need to give rigorous proofs, but please show your work (or explain your reasoning) in enough detail that I can understand and check your steps.

   (a) (1 point) Let X be the product of a Mobius band M and a cylinder C.

   (b) (2 points) Fix $g \geq 1$ and a point x_0 in the genus-g surface Σ_g. Let X be the once-punctured surface $\Sigma_g \setminus \{x_0\}$.

   (c) (2 points) The space X is the 3-skeleton of the product $\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2$ (with the usual CW structure).

(d) (3 points) Let X be the following polygons modulo the indicated edge identifications.



3. (a) (1 point) Let $\mathcal{C}$ be a category. State the universal property of the coproduct of two objects of $\mathcal{C}$.

(b) (1 point) Let $\underline{\mathrm{Top}_*}$ be the category of based spaces (X, x_0) and basepoint-preserving continuous maps. What is the coproduct of objects (X, x_0) and (Y, y_0), along with the associated maps? **No justification needed.**

4. (7 points) For each of the following statements: if the statement is true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Consider a space X. If a subspace $A \subseteq X$ is a retract of X, then A is a deformation retract of X.

   (b) Let $f : X \to Y$ be a continuous map of topological spaces. If its domain X is contractible, then its image $f(X)$ is a contractible subspace of Y.

   (c) There does not exist a continuous surjective map from the infinite earring to a CW complex obtained by taking the wedge $\bigvee_{\mathbb{N}} S^1$ of countably many circles.

(d) Let X be a contractible space. Then X admits a CW complex structure.

(e) Let F be a functor from the category Top of topological spaces and continuous maps, to the category Grp of groups and group homomorphisms. Then F must map the one-point space $*$ to the trivial group.

(f) There does not exist a homotopy equivalence between a 3-torus $T^3 = S^1 \times S^1 \times S^1$ and a genus-3 surface Σ_3.

(g) Let X be a contractible space, and let Y be a space obtained from X by gluing a collection of 2-disks along their boundaries via continuous attaching maps. Then necessarily $\pi_1(Y) \cong 0$.