Midterm Exam II

Math 592
26 March 2025
Jenny Wilson

Name: ____________________

Instructions: This exam has 5 questions for a total of 16 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 90 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Question	Points	Score
1	4	
2	4	
3	4	
4	3	
5	1	
Total:	16	

Notation

- $I = [0,1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$
  (unit n-sphere)
  (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ complex projective n-space

1. (4 points) Let $n \geq 2$. Let S^n denote the n-sphere, and let $x_0 \in S^n$ be a fixed basepoint. Let T denote the torus, and fix a basepoint $y_0 \in T$. Give an explicit proof that every based map

$$f : (S^n, x_0) \to (T, y_0)$$

is nullhomotopic via a *based* homotopy, i.e., a homotopy stationary on x_0.

Please include a complete statement of any theorems from our course that you cite.

Problem 1 continued.

2. Construct the following. No formal proof needed, but please include enough details of your thought process to let me verify your solution.
   (a) (2 points) Generators for a subgroup H of the free group F_2 on $\{a, b\}$ such that $H \subseteq N(H)$ is index-2, and $N(H) \subsetneq F_2$.

   (b) (2 points) Generators for a subgroup H of the free group F_3 on $\{a, b, c\}$ such that $H \subseteq F_3$ has index 3 and $N(H) = H$.

3. (a) (1 point) Let $p : \widetilde{X} \to X$ be a covering space map, and let $x_0 \in X$ be a point in its image. State the definition of the action of $\pi_1(X, x_0)$ on the fibre $p^{-1}(x_0)$.

(b) (3 points) Show that two points $\widetilde{x_1}, \widetilde{x_0} \in p^{-1}(x_0)$ are in the same path-component of $\widetilde{X}$ if and only if they are in the same orbit under the action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$.

4. (3 points) Let $n \geq 1$ and $0 \leq k \leq n-1$. Let S^n denote the n-sphere. Show that there cannot exist a retraction from S^n to any subspace A of S^n homeomorphic to S^k. In particular, the equator $S^{n-1} \subseteq S^n$ is not a retract of S^n.

5. (1 point) State the homology of the chain complex

$$\mathbb{Z}^4 \xrightarrow{\begin{bmatrix} 1791 & 4443 & 10074 & 15102 \\ 990 & 2450 & 5530 & 8250 \\ 687 & 1705 & 3869 & 5805 \\ 1632 & 4054 & 9215 & 13851 \\ 2235 & 5545 & 12575 & 18855 \\ 531 & 1317 & 2985 & 4473 \end{bmatrix}} \mathbb{Z}^6 \xrightarrow{\begin{bmatrix} 827 & -831 & 2498 & -3375 & 7783 & -26858 \\ -712 & 714 & -2148 & 2900 & -6686 & 23078 \\ 846 & -846 & 2544 & -3426 & 7884 & -27222 \end{bmatrix}} \mathbb{Z}^3$$

The differentials have Smith normal forms $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \end{bmatrix}$.