Midterm Exam II

Math 592
26 March 2025
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Name: ______________________________

Instructions: This exam has 5 questions for a total of 16 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 90 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Question	Points	Score
1	4	
2	4	
3	4	
4	3	
5	1	
Total:	16	

Notation

- $I = [0,1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ (unit n-sphere) (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{RP}^n$ real projective n-space
- $\mathbb{CP}^n$ complex projective n-space

1. (4 points) Let $n \geq 2$. Let S^n denote the n-sphere, and let $x_0 \in S^n$ be a fixed basepoint. Let T denote the torus, and fix a basepoint $y_0 \in T$. Give an explicit proof that every based map

$$f : (S^n, x_0) \to (T, y_0)$$

is nullhomotopic via a *based* homotopy, i.e., a homotopy stationary on x_0.

Please include a complete statement of any theorems from our course that you cite.

Proof. We can identify the torus with the orbit space $\mathbb{R}^2/\mathbb{Z}^2$ under the action of $\mathbb{Z}^2$ by translation. Then the quotient map $p : \mathbb{R}^2 \to T$ is the universal covering map, as in Homework #8 Problem 1. Let $r_0 \in p^{-1}(y_0)$ be any choice of preimage of the basepoint. We will invoke the *lifting criterion* for covering spaces,

> **Theorem.** Suppose $p : (\widetilde{Y}, r_0) \to (Y, y_0)$ is a covering space map, and suppose $f : (X, x_0) \to (Y, y_0)$ a based map with X path-connected and locally path-connected. Then a lift $\widetilde{f} : (X, x_0) \to (\widetilde{Y}, r_0)$ of f exists if and only if $f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(\widetilde{Y}, r_0))$.

Since S^n is a connected CW complex, it is path-connected and locally path-connected. We proved (since S^n admits a CW complex structure with no 1-cells) that $\pi_1(S^n) \cong 0$. Therefore the hypothesis on fundamental groups is vacuously satisfied. Thus we have a map

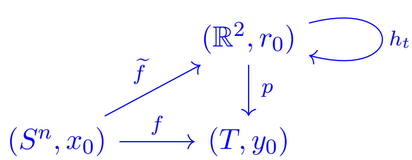
$$\widetilde{f} : (S^n, x_0) \to (\mathbb{R}^2, r_0)$$

such that $p \circ \widetilde{f} = f$. Let $h_t : \mathbb{R}^2 \to \mathbb{R}^2$ be straight-line homotopy from $id_{\mathbb{R}^2}$ to the constant map at r_0,

$$h_t(r) = r_0 + (1-t)(r - r_0)$$

Notably, $h_t(r_0) = r_0$ for all $t \in I$. Then define

$$f_t(x) = p \circ h_t \circ \widetilde{f}(x).$$



Then when $t = 0$,

$$f_0(x) = p \circ h_0 \circ \widetilde{f}(x) = p \circ id_{\mathbb{R}} \circ \widetilde{f}(x) = f(x) \qquad \text{for all } x \in X.$$

When $t = 1$,

$$f_1(x) = p\left(h_1\left(\widetilde{f}(x)\right)\right) = p(r_0) = y_0 \qquad \text{for all } x \in X.$$

When $x = x_0$,

$$f_t(x_0) = p \circ h_t \circ \widetilde{f}(x_0) = p \circ h_t(r_0) = p(r_0) = y_0 \qquad \text{for all } t \in I.$$

Thus f_t is a basepoint-preserving nullhomotopy as desired.

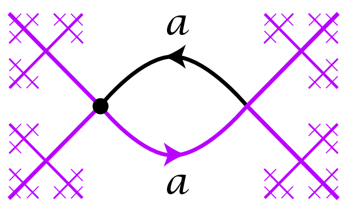
Problem 1 continued.

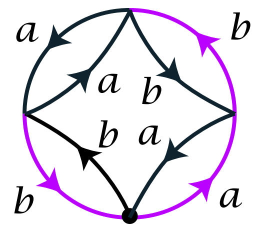
2. Construct the following. No formal proof needed, but please include enough details of your thought process to let me verify your solution.

   (a) (2 points) Generators for a subgroup H of the free group F_2 on $\{a, b\}$ such that $H \subseteq N(H)$ is index-2, and $N(H) \subsetneq F_2$.

   Proof. To construct H we construct the corresponding connected cover p of the graph $S^1 \vee S^1$ with single vertex v_0. The statement that $H \subseteq N(H)$ has index 2 means that the deck group $N(H)/H$ must be order 2, and the statement that $N(H) \neq F_2$ means that the cover is not normal, that is, the deck group is non-transitive on $p^{-1}(v_0)$. This means the degree of p must be greater than 2.

   Two examples of possible covers are shown (with a choice of maximal tree highlighted in purple). The single non-trivial deck transformation is given by 180° rotation in both cases.

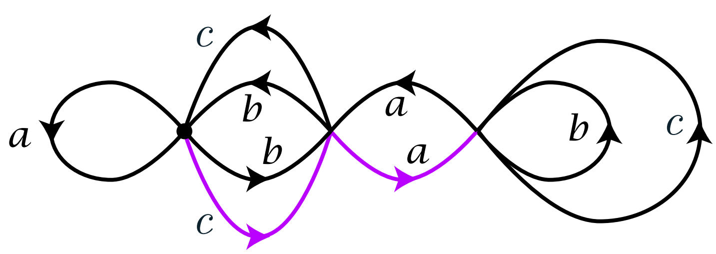




   The first corresponds to the subgroup $\langle a^2 \rangle \subseteq F_2$, and the second to the subgroup $\langle a^2, b^2, (ab)^2, ab^2a, b^{-1}ab^{-1}a^{-1} \rangle \subseteq F_2$, as determined using the given choices of maximal trees.

   (b) (2 points) Generators for a subgroup H of the free group F_3 on $\{a, b, c\}$ such that $H \subseteq F_3$ has index 3 and $N(H) = H$.

   Proof. We will construct the corresponding connected cover p of the graph $S^1 \vee S^1 \vee S^1$ with single vertex v_0. The statement that $H = N(H)$ means that the cover must have no nontrivial deck transformations, and the condition that $H \subseteq F_3$ has index 3 means that the cover is degree-3, that is, its vertex set $p^{-1}(v_0)$ has three elements.



   One possible example is shown, with a choice of maximal tree highlighted in purple. With this choice of maximal tree, we determine that the image of its fundamental group in $\pi_1(S^1 \vee S^1 \vee S^1, v_0) \cong F_3$ is

$$H = \langle a, c^2, cb, bc^{-1}, ca^2c^{-1}, (ca)b(ca)^{-1}, (ca)c(ca)^{-1} \rangle \subseteq F_3$$

3. (a) (1 point) Let $p : \widetilde{X} \to X$ be a covering space map, and let $x_0 \in X$ be a point in its image. State the definition of the action of $\pi_1(X, x_0)$ on the fibre $p^{-1}(x_0)$.

Definition. Given a class $[\alpha] \in \pi_1(X, x_0)$ represented by a loop α, we define an associated permutation on $p^{-1}(x_0)$ as follows. For a point $\widetilde{x_0} \in p^{-1}(x_0)$, we consider the (uniquely defined) lift $\widetilde{\overline{\alpha}}$ of the inverse loop $\overline{\alpha}$ to a path starting at $\widetilde{x_0}$. Then $[\alpha]$ maps $\widetilde{x_0}$ to the endpoint $\widetilde{\overline{\alpha}}(1) \in p^{-1}(x_0)$ of this path.

You proved on Homework #7 Problem 2 that this permutation does not depend on the choice of representative loop α, and that it defines a group action of $\pi_1(X, x_0)$ on the fibre $p^{-1}(x_0)$.

(b) (3 points) Show that two points $\widetilde{x_1}, \widetilde{x_0} \in p^{-1}(x_0)$ are in the same path-component of $\widetilde{X}$ if and only if they are in the same orbit under the action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$.

Lemma. Let $\beta : I \to X$ be a path in X from x_0 to x_1, and let $\widetilde{\beta} : I \to \widetilde{X}$ be its (unique) lift to $\widetilde{X}$ starting at a chosen point $\widetilde{x_0} \in p^{-1}(x_0)$. Let $\widetilde{x_1} = \widetilde{\beta}(1)$. Then the inverse path $\overline{\widetilde{\beta}}$ is the (unique) lift of the inverse path $\overline{\beta}$ starting at $\widetilde{x_1}$.

Proof of lemma. We know that, since the interval I is a connected space, the lift of any path $I \to X$ to $\widetilde{X}$ is determined by the image of a single point, in particular it is determined by its starting point, the image of 0.

By definition, $\overline{\beta}(t) = \beta(1-t)$ and $\overline{\widetilde{\beta}}(t) = \widetilde{\beta}(1-t)$, and $p \circ \widetilde{\beta} = \beta$. Then,

$$p \circ \overline{\widetilde{\beta}}(t) = p \circ \widetilde{\beta}(1-t) = \beta(1-t) = \overline{\beta}(t).$$

This confirms that $\overline{\widetilde{\beta}}$ is indeed a lift of $\overline{\beta}(t)$, and so it must be the unique lift starting at $\widetilde{x_1}$.

Proof of problem. Suppose that $\widetilde{x_1}, \widetilde{x_0} \in p^{-1}(x_0)$ are in the same path component of $\widetilde{X}$. This means, by definition, that there exists a path $\gamma : I \to \widetilde{X}$ from $\widetilde{x_1}$ to $\widetilde{x_0}$. Then $\alpha := p \circ \gamma$ is a loop in X based at x_0, representing an element $[\alpha]$ in $\pi_1(X, x_0)$. By construction, γ is the (unique) lift of α starting at $\widetilde{x_1}$. By the lemma, its inverse path $\overline{\gamma}$ is the unique lift of $\overline{\alpha}$ starting at $\widetilde{x_0}$. Thus $[\alpha]$ maps $\widetilde{x_0}$ to $\widetilde{x_1}$, and we conclude that these points are in the same orbit.

Conversely, suppose that some $[\alpha] \in \pi_1(X, x_0)$ maps some $\widetilde{x_0} \in p^{-1}(x_0)$ to $\widetilde{x_1} \in p^{-1}(x_0)$. This means that $\overline{\alpha}$ lifts to a path from $\widetilde{x_0}$ to $\widetilde{x_1}$, so these points must be in the same path component of $\widetilde{X}$.

4. (3 points) Let $n \geq 1$ and $0 \leq k \leq n-1$. Let S^n denote the n-sphere. Show that there cannot exist a retraction from S^n to any subspace A of S^n homeomorphic to S^k. In particular, the equator $S^{n-1} \subseteq S^n$ is not a retract of S^n.

Proof. Suppose (for the sake of contradiction) that $r : S^n \to A$ were a retraction. This means, by definition, the composite with the inclusion map $\iota : A \to S^n$ satisfies

$$A \xrightarrow{\iota} S^n \xrightarrow{r} A, \qquad r \circ \iota = id$$

We proved in class that $\widetilde{H}_i(S^r) \cong \begin{cases} \mathbb{Z}, & i = r \\ 0, & i \neq r. \end{cases}$

Since degree-k reduced homology $\widetilde{H}_k$ is functorial, it follows that we have a commutative diagram

$$\begin{array}{ccccc} \widetilde{H}_k(A) & \xrightarrow{\iota_*} & \widetilde{H}_k(S^n) & \xrightarrow{r_*} & \widetilde{H}_k(A). \\ \shortparallel & & \shortparallel & & \shortparallel \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array} \qquad r_* \circ \iota_* = id$$

This is contradiction; the identity map on the nonzero group $\widetilde{H}_k(A) \cong \mathbb{Z}$ cannot factor through the trivial group. We conclude that no retraction $r : S^n \to A$ can exist.

5. (1 point) State the homology of the chain complex

$$\mathbb{Z}^4 \xrightarrow{\begin{bmatrix} 1791 & 4443 & 10074 & 15102 \\ 990 & 2450 & 5530 & 8250 \\ 687 & 1705 & 3869 & 5805 \\ 1632 & 4054 & 9215 & 13851 \\ 2235 & 5545 & 12575 & 18855 \\ 531 & 1317 & 2985 & 4473 \end{bmatrix}} \mathbb{Z}^6 \xrightarrow{\begin{bmatrix} 827 & -831 & 2498 & -3375 & 7783 & -26858 \\ -712 & 714 & -2148 & 2900 & -6686 & 23078 \\ 846 & -846 & 2544 & -3426 & 7884 & -27222 \end{bmatrix}} \mathbb{Z}^3$$

The differentials have Smith normal forms $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \end{bmatrix}$.

Solution. Call the matrices A and B, respectively. By Homework #8 Problem 5(c), the homology is

$$\ker(B)/\mathrm{im}(A) \quad \cong \quad \mathbb{Z}^{6-\mathrm{rank}(A)-\mathrm{rank}(B)} \oplus \bigoplus_{\substack{\text{invariant factors} \\ \alpha \text{ of } A}} \mathbb{Z}/\alpha\mathbb{Z} \quad \cong \quad \mathbb{Z} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}.$$