Midterm Exam I

Math 592
12 February 2025
Jenny Wilson

Name: ____

Instructions: This exam has 4 questions for a total of 20 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 90 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Question	Points	Score
1	3	
2	8	
3	2	
4	7	
Total:	20	

Notation

- $I = [0,1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$
  (unit n-sphere)
  (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n\geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{RP}^n$ real projective n-space
- $\mathbb{CP}^n$ complex projective n-space

1. (3 points) Let X be a space, and let $f : S^1 \to X$ be a continuous map. Prove that the following are equivalent. *Hint:* You can use without proof the fact that the quotient space $(S^1 \times I)/(S^1 \times \{1\})$ is a 2-disk.

   (i) View the domain of f as the boundary of the closed 2-disk D^2. The map f extends to a continuous map $D^2 \to X$.

   (ii) The induced map $f_* : \pi_1(S^1, s) \to \pi_1(X, f(s))$ is the zero map (with respect to any choice of basepoint $s \in S^1$).

Proof that (i) $\implies$ (ii).

Suppose that the map f extends over the disk. This means that we can express f as the composite

$$S^1 = \partial D^2 \hookrightarrow D^2 \longrightarrow X \qquad (\text{composite } f)$$

Choose a basepoint $s \in S^1$. Because the fundamental group is functorial, it follows that the induced map f_* factors through $\pi_1(D^2, s)$.

$$\pi_1(\partial D^2, s) \longrightarrow \pi_1(D^2, s) \longrightarrow \pi_1(X, f(s)) \qquad (\text{composite } f_*)$$

But the disk D^2 is contracible, hence $\pi_1(D^2, s) = 0$. Thus f_* is the zero map.

Proof that (ii) $\implies$ (i).

View S^1 as the quotient of the closed interval I modulo its boundary $\partial I = \{0, 1\}$, and let the basepoint s be the image of the boundary.

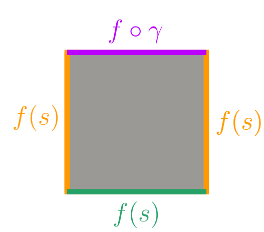
Suppose that $f_* : \pi_1(S^1, s) \to \pi_1(X, f(s))$ is the zero map. This means, for any loop $\gamma : I \to S^1$ based at s, its image

$$f_*([\gamma]) = [f \circ \gamma]$$

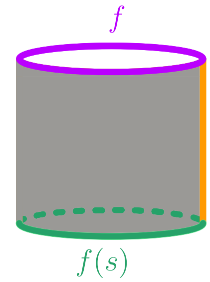
will be zero in $\pi_1(X, f(s))$. Let $\gamma : I \to S^1 = I/\partial I$ be the quotient map. Let

$$\begin{aligned} F : I \times I &\longrightarrow X \\ (a, t) &\longmapsto X \end{aligned}$$

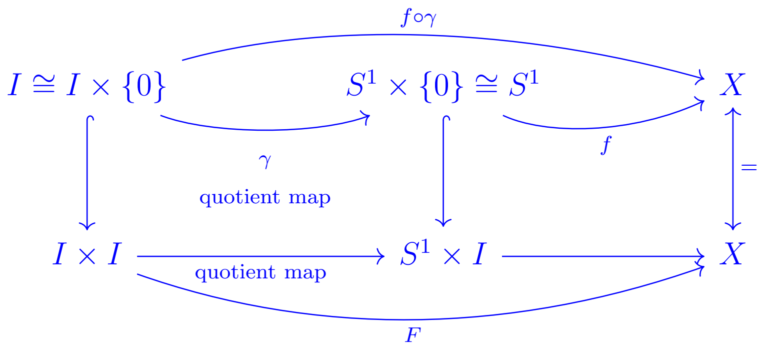
be a nullhomotopy of $f \circ \gamma$, with $F(0, t) = (f \circ \gamma)(t)$ and $F(1, t)$ the constant map at the point $f(s) \in Y$. The domain of F is pictured.



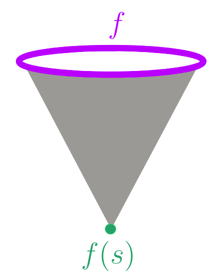
Since at each fixed time a_0 the map $F(a_0, t)$ is a loop based at $f(s)$, by the universal property of the quotient topology, the map F factors uniquely through a map from the cylinder $I \times S^1$, the quotient space of $(I \times I)$ modulo the equivalence relation $(a_0, 0) \sim (a_0, 1)$ for each $a_0 \in I$.



Consider the restriction of the induced map on the cylinder to the 'top' boundary component of the cylinder, the circle $S^1 \times \{0\} \cong S^1$. This is the map induced by $f \circ \gamma$ on the quotient $I/\partial I \cong S^1$. The universal property of the quotient topology implies $f \circ \gamma$ factors uniquely through the quotient, hence the induced map must be f, as labelled in purple on the diagram.



Because the induced map on the cylinder is constant on $\{1\} \times S^1$, the map further factors through the quotient $(I \times S^1)/(\{1\} \times S^1)$. This quotient is homeomorphic to a 2-disk, and the restriction of the map to the boundary is f.



Thus, we have proved that f extends over D^2.

2. For each of the following spaces X, compute the fundamental group.

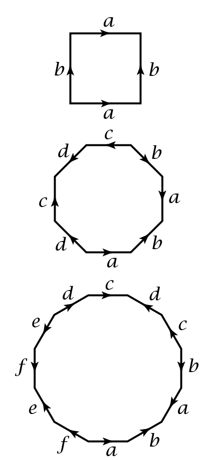
   You do not need to give rigorous proofs, but please show your work (or explain your reasoning) in enough detail that I can understand and check your steps.

   (a) (1 point) Let X be the product of a Mobius band M and a cylinder C.

   Solution. $\pi_1(M \times C) \cong \mathbb{Z}^2$.
   We know
   $$\pi_1(M \times C) \cong \pi_1(M) \times \pi_1(C)$$
   by our result from class on the fundamental group of a product. Then
   $$\pi_1(M) \times \pi_1(C) \cong \pi_1(S^1) \times \pi_1(S^1)$$
   by our proof from class that M and C deformation retract onto their equatorial circles, and the result from Homework #3 Problem 4(d) that π_1 is a homotopy invariant. Finally,
   $$\pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z}^2$$
   by the calculation done in Homework #2 Problem 2.

   (b) (2 points) Fix $g \geq 1$ and a point x_0 in the genus-g surface Σ_g. Let X be the once-punctured surface $\Sigma_g \setminus \{x_0\}$.

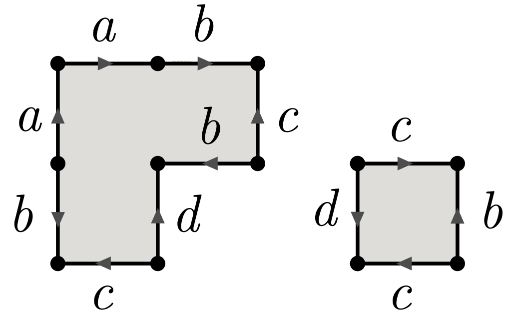
   Solution. $\pi_1(\Sigma_g \setminus \{x_0\}) \cong F_{2g}$, the free group on $2g$ generators.
   View the genus g surface Σ_g as a $4g$-gon with edge identifications, as on Homework #4 Problem 2(a), pictured for $g = 1, 2, 3$. If we choose x_0 to be the center point of the polygon, then we can deformation retract the punctured polygon onto its boundary, using a straight-line homotopy along lines emanating radially from x_0. This deformation retraction induces a well-defined deformation retraction on the quotient space by the edge identifications, by our criterion for homotopies of quotient spaces (Homework #1 Problem 3.) But the boundary with edge identifications is a wedge of $2g$ circles, and so its fundamental group is the free group on $2g$ generators, by our calcuation of $\pi_1(S^1)$ and our result that the fundamental group of a wedge sum is the free product of the fundamental groups of the factors.



   (c) (2 points) The space X is the 3-skeleton of the product $\mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{RP}^2$ (with the usual CW structure).

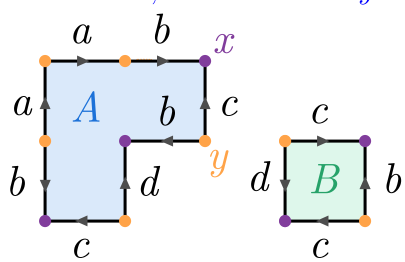
   Solution. $\pi_1(X) \cong (\mathbb{Z}/2\mathbb{Z})^3$.
   Observe that the 2-skeleton X^2 of X is also the 2-skeleton of $\mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{RP}^2$. But we proved on Homework #4 Problem 2 that the inclusion of the 2-skeleton of a CW complex induces an isomorphism on fundamental group. Hence $\pi_1(X) \cong \pi_1(X^2) \cong \pi_1(\mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{RP}^2)$. The result then follows from the computation of $\pi_1(\mathbb{RP}^2)$ (Homework #4 Problem 3(a)) and our result on the fundamental group of a product.

(d) (3 points) Let X be the following polygons modulo the indicated edge identifications.

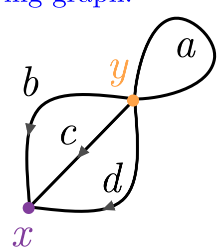


Solution. $\pi_1(X) \cong \langle a, b, c \mid abc^{-1}bcb^{-1}a, cb^{-1}c \rangle$.

Label the two 2-disks A and B as shown. When we trace through the edge identifications, we find that X has two distinct vertices, labelled x and y.



Thus its 1-skeleton X^1 is the following graph.



To find free generators of $\pi_1(X^1)$, we choose a maximal tree, say, the edge d. Since this edge is contractible and embeds into X, the quotient X/d of X by the edge d is a homotopy equivalence by Homework #2 Problem 2. Since fundamental group is a homotopy invariant (Homework #3 Problem 4(d)), it is equivalent, up to group isomorphism, to compute the fundamental group of X/d.

In this quotient space, the fundamental group of the 1-skeleton is freely generated by the loops a, b, c. The 2-disk A is glued along the word $abc^{-1}bcb^{-1}a$ (well-defined up to cyclic permutation), and the 2-disk B is glued along the word $cb^{-1}c$ (well-defined up to cyclic permutation). By our procedure for computing the fundamental group of a CW complex (Homework #4 Problem 1), we obtain the presentation $\langle a, b, c \mid abc^{-1}bcb^{-1}a,\ cb^{-1}c \rangle$.

3. (a) (1 point) Let $\mathcal{C}$ be a category. State the universal property of the coproduct of two objects of $\mathcal{C}$.

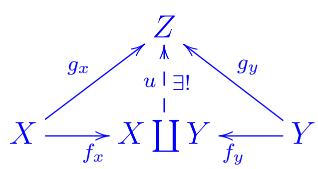
Solution. Let $\mathcal{C}$ be a category with objects X and Y. The *coproduct* of X and Y (if it exists) is an object $X \coprod Y$ in $\mathcal{C}$ with morphisms

$$f_x : X \to X \coprod Y, \qquad \text{and} \qquad f_y : Y \to X \coprod Y$$

satisfying the following universal property: whenever there is an object Z with morphisms

$$g_x : X \to Z \qquad \text{and} \qquad g_y : Y \to Z,$$

there exists a unique morphism $u : X \coprod Y \to Z$ that makes the following diagram commute:



(b) (1 point) Let $\underline{\text{Top}_*}$ be the category of based spaces (X, x_0) and basepoint-preserving continuous maps. What is the coproduct of objects (X, x_0) and (Y, y_0), along with the associated maps? **No justification needed.**

Solution. Let $X \vee Y$ be the wedge sum of X and Y obtained by identifying their basepoints x_0 and y_0. By viewing the wedge sum $X \vee Y$ as a quotient space of the disjoint union $X \sqcup Y$, we can denote the wedge point as the equivalence class $\{x_0, y_0\} \subseteq X \sqcup Y$. Then the coproduct $(X, x_0) \coprod (Y, y_0)$ is the based space

$$(X \vee Y, \{x_0, y_0\})$$

along with the based maps

$$(X, x_0) \to (X \vee Y, \{x_0, y_0\}) \qquad \text{and} \qquad (Y, y_0) \to (X \vee Y, \{x_0, y_0\})$$

obtained by restricting the quotient map $X \sqcup Y \to X \vee Y$ to X and to Y, respectively.

Hint: To verify that this construction satisfies the universal property, use the universal properties of the disjoint union and the quotient topology, respectively, to construct the morhpism u and verify that it is well-defined and unique.

4. (7 points) For each of the following statements: if the statement is true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Consider a space X. If a subspace $A \subseteq X$ is a retract of X, then A is a deformation retract of X.

   False. For example, for any space X, any one-point subspace $\{x\} \subseteq X$ is a retract of X—constant maps are retractions—but a space will never deformation retract to a point if it is noncontractible. So, specifically, the subspace $\{1\} \subseteq S^1$ is a retract but not a deformation retract.

   Other examples of $A \subseteq X$ are: $S^1 \times \{1\} \subseteq S^1 \times S^1$, one closed hemisphere of S^n, one copy of S^1 in $S^1 \vee S^1$, one copy of D^2 in the disjoint union $D^2 \sqcup D^2$, .

   (b) Let $f : X \to Y$ be a continuous map of topological spaces. If its domain X is contractible, then its image $f(X)$ is a contractible subspace of Y.

   False. The hypotheses imply that the map f is nullhomotopic, but this does not that the image must be contractible as a subspace of Y. Consider, for example, the (surjective) quotient map $I \to S^1$ that identifies the two boundary points of the closed interval I.

   Other counterexamples: the quotient map $D^n \to S^n$, the covering map $\mathbb{R} \to S^1$.

   (c) There does not exist a continuous surjective map from the infinite earring to a CW complex obtained by taking the wedge $\bigvee_{\mathbb{N}} S^1$ of countably many circles.

   True.

   Hint: The infinite earring is compact, but a CW complex with infinitely many cells is non-compact (Homework #1 Problem 8(b)). The continuous image of a compact space is compact.

   (There does, however, exist a continuous surjective map f the other way, mapping each circle in the wedge to a circle of the infinite earring. This is consistent with the result of your Homework #1 Problem 8(e), where you proved that f is not a homeomorphism.)

(d) Let X be a contractible space. Then X admits a CW complex structure.

False. You proved on Homework #2 Problem 2(g) that the two-point space $X = \{a, b\}$ with the topology $\{\varnothing, \{a\}, X\}$ is contractible. But X does not a have a CW complex structure; any CW complex consisting of finitely many points must have the discrete topology. In particular X is not Hausdorff, violating Homework #1 Problem 7.

Other counterexamples: The cone on the infinite earring, the subset of $\mathbb{R}^2$ consisting of 0 and a radial ray at each rational multiple of 2π, the subset of $\mathbb{R}^2$ that is the union of the x-axis and the vertical lines $\left\{\frac{1}{n}\right\} \times \mathbb{R}$ for $n \in \mathbb{N}$, any space with at least two points and the indiscrete topology.

(e) Let F be a functor from the category Top of topological spaces and continuous maps, to the category Grp of groups and group homomorphisms. Then F must map the one-point space $*$ to the trivial group.

False. Let G be a nontrivial group (say, $G = \mathbb{Z}/2\mathbb{Z}$). Consider the 'constant functor' at G, which maps every object to G, and every morphism to the identity map of G.

Other counterexample: The composition of the forgetful map Top $\to$ Set and the free functor Set $\to$ Grp will map $*$ to $\mathbb{Z}$.

(f) There does not exist a homotopy equivalence between a 3-torus $T^3 = S^1 \times S^1 \times S^1$ and a genus-3 surface Σ_3.

True.

Hint: These spaces have non-isomorphic fundamental groups. The 3-torus has $\pi_1(T^3) \cong \mathbb{Z}^3$ (by our result for fundamental group of S^1 and fundamental groups of products), and $\pi_1(\Sigma_3)$ has presentation $\langle a_1, b_1, a_2, b_2, a_3, b_3 \mid [a_1, b_1][a_2, b_2][a_3, b_3] \rangle$. In particular, the abelianization of $\pi_1(T^3)$ is $\mathbb{Z}^3$ and the abelianization of $\pi_1(\Sigma_3)$ is $\mathbb{Z}^6$, which are nonisomorphic by the classification of finitely generated abelian groups.

(g) Let X be a contractible space, and let Y be a space obtained from X by gluing a collection of 2-disks along their boundaries via continuous attaching maps. Then necessarily $\pi_1(Y) \cong 0$.

True.

Hint: This is an application of Homework #4 Problem 1(a).