Notation

- $I = [0, 1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ (n-sphere) (we sometimes view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ real complex n-space

Practice problems

1. Let X be a topological space, and let $f, g : X \to S^n$ be two maps with the property that the points $f(x)$ and $g(x)$ are never antipodal for any $x \in X$. Prove that f and g are homotopic.
2. Suppose that a map $f : S^1 \to S^1$ is nullhomotopic. Show that f has a fixed point, and maps at least one point x to its antipode $-x$.
3. Let X and Y be topological spaces.
   (a) Suppose that Y is contractible. Prove that any two maps from X to Y are homotopic.
   (b) Suppose that X is contractible and Y is path-connected. Show that any two maps from X to Y are homotopic.
4. (a) Show that a homotopy equivalence $f : X \to Y$ induces a bijection between the set of path-components of X and the set of path-components of Y .
   (b) Show that f restricts to a homotopy equivalence from each path-component of X to the corresponding path-component of Y .
5. A topological space G with a group structure is called a *topological group* if the group multiplication map and inverse map

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto g \cdot h \end{aligned} \qquad\qquad \begin{aligned} G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

   are continuous. Suppose G is a path-connected topological group. Show that, for each $g_0 \in G$, the left multiplication map

$$\begin{aligned} \ell_{g_0} : G &\longrightarrow G \\ g &\longmapsto g_0 \cdot g \end{aligned}$$

   is homotopic to the identity.
6. Let X be a CW complex. Show that any finite collection of cells in X are contained in a finite subcomplex.
7. Give an example (with proof) of a topological space that does not admit a CW complex structure.
8. **(QR Exam, January 2024)**.
   (a) State the definition of a CW complex, and its topology (the weak topology).
   (b) Let X be a CW complex and $A \subseteq X$ a nonempty CW subcomplex. Working directly from your definition, describe a CW complex structure on the quotient space X/A, and verify explicitly that the quotient topology on X/A agrees with the weak topology of your given CW complex structure.
9. Let X be a CW complex with n_d cells in dimension d, and let Y be a CW complex with m_d cells in dimension d. We described a natural CW complex structure on $X \times Y$. Count its cells in dimension d.

10. Let (X, x_0) and (Y, y_0) be path-connected CW complexes. Let $f : (X, x_0) \to (X', x'_0)$ be a based map, which is a homotopy equivalence via homotopies stationary on x_0 and x'_0. Show that there is a homotopy equivalence between $X \vee Y$ and $X' \vee Y$. Here the wedge sums are formed by identifying the base points.

11. **Definition (Isomorphism).** Let $\mathscr{C}$ be a category. A morphism $f : X \to Y$ in $\mathscr{C}$ is called an *isomorphism* if there is a morphism $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$. In this case, we write $g = f^{-1}$ and call g the *inverse* of f.

    (a) Suppose a morphism $f : X \to Y$ in a category $\mathscr{C}$ has an inverse g. Verify that the inverse is unique (so calling g "the" inverse is justified.)

    (b) Show that an isomorphism is both monic and epic.

    (c) Show that a map can be monic and epic, but not an isomorphism. *Hint:* Consider a category with only two objects.

    (d) Let $f : X \to Y$ be a morphism in a category $\mathscr{C}$. Suppose there were morphisms $g, h : Y \to X$ such that $f \circ g = id_Y$ and $g \circ h = id_X$. Show that $g = h$, and conclude that f is an isomorphism.

    (e) Let $f : X \to Y$ be a map of topological spaces. Suppose there exists maps $g, h : Y \to X$ so that $f \circ g$ and $h \circ f$ are both homotopic to identity maps. Prove that f is a homotopy equivalence.

12. Show that a map homotopic to a homotopy equivalence is itself a homotopy equivalence.

13. Let $\underline{\text{Set}}$ be the category of sets, and for a set A let $\text{Hom}_{\underline{\text{Set}}}(-, A)$ denote the associated contravariant hom functor

$$\begin{aligned} \text{Hom}_{\underline{\text{Set}}}(-, A) : \mathscr{C} &\longrightarrow \underline{\text{Set}} \\ B &\longmapsto \text{Hom}_{\underline{\text{Set}}}(B, A) \\ [f : B \to C] &\longmapsto \left[f^* : \begin{array}{rl} \text{Hom}_{\underline{\text{Set}}}(C, A) & \to \text{Hom}_{\underline{\text{Set}}}(B, A)] \\ \phi & \mapsto \phi \circ f \end{array} \right] \end{aligned}$$

    Prove that, if f is surjective, then f^* is injective.

14. Let $\underline{\text{Top}}_*$ be the category of based topological spaces and based maps. What is the coproduct of based spaces X and Y, along with the two associated maps? Prove your answer.

15. Let $\underline{\text{Top}}$ be the category of topological spaces, and let $P : \underline{\text{Top}} \to \underline{\text{Set}}$ be the map that takes a topological space X to its set of path components. Explain how to define P on morphisms to make it a functor, and verify that it is well-defined and functorial.

16. Recall that we defined a forgetful map

$$\begin{aligned} \underline{\text{Top}}_* &\longrightarrow \underline{\text{Top}} \\ (X, x_0) &\longmapsto X \\ f &\longmapsto f \end{aligned}$$

    What is the "free functor" associated to this "forgetful functor"? For a topological space X, determine what universal property the "free based space on X" should satisfy, and describe what this topological space $F(X)$ (along with its basepoint, and distinguished map $X \to F(X)$) should be.

17. **Definition (Initial objects, terminal objects, zero objects).** An *initial object* I in a category $\mathscr{C}$, if it exists, is an object with the property that for any $X \in \mathscr{C}$, there is a unique morphism in $\mathscr{C}$ from I to X. Dually, an object T is a *terminal object* if for every $X \in \mathscr{C}$ there is a unique morphism $X \to T$. If an object is both initial and terminal, it is called a *zero object*.

    (a) Show that, if an initial (or terminal, or zero) object exists in a category $\mathscr{C}$, it is unique up to unique isomorphism.

(b) Determine whether initial, terminal, or zero objects exists, and what they are, in the categories $\underline{\text{Set}}$, $\underline{\text{Ab}}$, $\underline{\text{Grp}}$, $\underline{\text{Top}}$, and $\underline{\text{Top}}_*$.

(c) Let $\mathscr{C}$ be a category and I an initial object in $\mathscr{C}$. Prove or disprove: If $F : \mathscr{C} \to \mathscr{D}$ is a covariant functor, then $F(I)$ is an initial object in $\mathscr{D}$.

18. Let F_S denote the free group on the set S. Suppose that S_1 and S_2 are finite sets. Show that, if S_1 and S_2 have different cardinalities, then F_{S_1} and F_{S_2} are not isomorphic.
Hint: Show that $\text{Hom}_{\underline{\text{Grp}}}(F_S, \mathbb{Q})$ has the structure of a $\mathbb{Q}$-vector space, and compute its dimension.

19. (a) Let G be a group and $[G, G]$ is commutator subgroup. Let $\phi : G \to G$ be an automorphism. Prove that $\phi([G, G]) \subseteq [G, G]$. (This means that $[G, G]$ is a *characteristic* subgroup.)

(b) Show that an automorphism $\phi : G \to G$ induces an automorphism $\tilde{\phi} : G^{ab} \to G^{ab}$.

(c) Let $\text{Aut}(G)$ denote the group of automorphisms of G. Prove that the induced map

$$\begin{aligned} \Phi : \text{Aut}(G) &\longrightarrow \text{Aut}(G^{ab}) \\ \phi &\longmapsto \tilde{\phi} \end{aligned}$$

is a group homomorphism.

20. (a) Let X be a space, and let x_0 and x_1 be two points in the same path component of X. Given a path h from x_0 to x_1, let β_h be the associated change-of-basepoint map

$$\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0).$$

Choose a (possibly different) path g from x_1 to x_0. Show that the automorphism

$$\beta_h \circ \beta_g : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

is given by conjugation by an element of $\pi_1(X, x_0)$ (which one?). Such automorphisms are called *inner automorphisms*.

(b) Explain the sense in which elements of $\pi_1(X, x_0)$ depend on the choice of basepoint, but conjugacy classes of elements in $\pi_1(X, x_0)$ do not.

(c) Suppose that $\pi_1(X, x_0)$ is abelian. Show that the isomorphism

$$\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$$

is independent of choice of path h.

(d) Suppose X is path-connected and $\pi_1(X, x_0)$ is abelian. Explain the sense in which a loop in X gives a well-defined element of $\pi_1(X, x)$ with respect to any basepoint x.

21. We showed that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ is generated by the class of a loop α around a meridian circle and the class of a loop a longitudinal circle, as in Figure 1. Construct an explicit homotopy from $\alpha \cdot \beta$ to $\beta \cdot \alpha$.

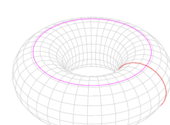


Figure 1: The loops α and β in Σ_1

22. (a) Suppose that $p_X : \tilde{X} \to X$ is a cover of a space X, and $p_Y : \tilde{Y} \to Y$ is a cover of a space Y. Show that $p_X \times p_Y : \tilde{X} \times \tilde{Y} \to X \times Y$ is a covering map.

(b) Conclude that $\mathbb{R}^n$ is a cover of the n-torus.

23. Suppose that $p_X : \tilde{X} \to X$ is a cover of a space X.
    (a) Show that, if X is locally Euclidean then so is $\tilde{X}$.
    (b) Show that, if X is Hausdorff, then so is $\tilde{X}$.

    With this problem, we are most of the way to showing that, if X is a topological manifold, then so is $\tilde{X}$. We will need an extra condition on $\tilde{X}$, however, to ensure it is second-countable.

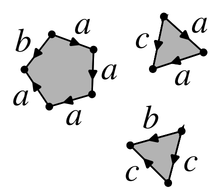
24. Let $f : X \to S^1$ be a continuous map. Show that, if f is nullhomotopic, then it factors through the covering map $p : \mathbb{R} \to S^1$.

25. Prove that a space X is simply-connected if and only if there is a unique homotopy class of paths connecting any two points in X .

26. Let $X \subseteq D^2$ be a subspace, and let $f : X \to X$ be a map without fixed points. Show that X is not a retract of D^2.

27. Show that there is no retraction from a Möbius band to its boundary.

28. Let X be the quotient space defined as the union of the polygons below, modulo the given edge identifications.



    (a) Compute a presentation for $\pi_1(X)$.
    (b) Prove that $\pi_1(X) \cong \mathbb{Z}$.
    (c) Let $B \subseteq X$ be the image of the loop b. Prove that B is not a retract of X.
    Hint: First show that the inclusion $4\mathbb{Z} \hookrightarrow \mathbb{Z}$ does not admit a one-sided inverse.

29. Let $S^1 \times I$ be the cylinder, and suppose that $f : S^1 \times I \to X$ is a map that is constant on $S^1 \times \{1\}$. Show that the map f_* induced by f on fundamental group is trivial.

30. (a) Let $C_n \subseteq \mathbb{R}^2$ be the circle of radius n and center $(n, 0)$. Let $X = \cup_n C_n$. Show that $\pi_1(X)$ is the free group on countably many generators, with n^{th} generator a loop around C_n.
    (b) Show that X is not homeomorphic to the infinite wedge $\bigvee_{n\in\mathbb{N}} S^1$.
    Hint: Show that the point $(0, 0)$ in X has a countable neighbouhood basis, but the wedge point of $\bigvee_{n\in\mathbb{N}} S^1$ does not.

31. Let G and H be groups. Prove that $(G * H)^{ab} = G^{ab} \oplus H^{ab}$.

32. (a) Let G be a group with presentation $\langle S \mid R\rangle$. Explain how to construct a 2-dimensional CW complex with fundamental group isomorphic to G.
    (b) Describe the construction of a CW complex with fundamental group is

$$\mathrm{SL}_2(\mathbb{Z}) \cong \langle a, b \mid abab^{-1}a^{-1}b^{-1}, (aba)^4\rangle.$$

33. Let X be a compact CW complex. Explain why $\pi_1(X)$ must be a finitely presented group.

34. (a) State the Cellular Approximation Theorem.

(b) Let $d \leq n-1$. Prove that every map from a d-dimensionl CW complex X to the n-sphere S^n is nullhomotopic.

35. **True or counterexample.** For each of the following statements: if the statement is true, write "True". If not, state a counterexample. No justification necessary.
Note: If the statement is false, you can receive partial credit for writing "False" without a counterexample.

(a) Any contractible space is path-connected.

(b) Any subspace of a contractible space is contractible.

(c) Any quotient of a contractible space is contractible.

(d) The product of two contractible spaces is contractible.

(e) For any topological space X, any two maps $X \to S^\infty$ are homotopic.

(f) For any topological space X, any two maps $\mathbb{R}^n \to X$ are homotopic.

(g) Let X be a space and A a subspace. If A is a retract of X, then A and X are homotopy equivalent. (Note the distinction between *retract* and *deformation retract*).

(h) A CW complex X is compact if and only if it is finite.

(i) Any compact subset of a CW complex is closed.

(j) Let F_S be the free group on a set S. Then every group isomorphism $F_S \to F_S$ is induced by a permutation of S.

(k) There exists no non-abelian group with abelianization $\mathbb{Z}$.

(l) Suppose a group G has a presentation with a single generator. Then G is abelian.

(m) If G is a finitely presented group, then its abelianization G^{ab} is finitely presented.

(n) Every topological space is homeomorphic to the topological disjoint union of its connected components.

(o) There does not exist a retraction from $\mathbb{R}^2$ to $\mathbb{R}^2 \setminus \{0\}$.

(p) There does not exist a retraction from the torus $S^1 \times S^1$ to its meridian $S^1 \times \{(1,0)\}$.

(q) Let $X \subseteq Y$ be a subspace, and $x_0 \in X$. Then $\pi_1(X, x_0)$ is a subgroup of $\pi_1(Y, x_0)$.

(r) Suppose that X is a union of open, contractible subsets. Then $\pi_1(X) = 0$.

(s) Any presentation of a finite group must be finite.

(t) If G is a finitely presented group, then there exists a compact Hausdorff space with fundamental group isomorphic to G.

(u) If X is a connected graph, then $\pi_1(X)$ is a free group.

36. Compute the fundamental groups of the following spaces, and give presentations. Draw or describe the generators as loops in the space. Recall some tools we have developed:

- fundamental group is a homotopy invariant
- fundamental groups of Cartesian products and wedge sums
- the quotient of a CW complex by a contractible subcomplex is a homotopy equivalence
- Van Kampen theorem
- our results from Homework #4 on the effect on π_1 of gluing in disks along their boundaries
- our results from Homework #4 on the fundamental groups of CW complexes

(a) The product of $\mathbb{R}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2$

(b) The wedge sum of a cylinder and a Möbius band

(c) A genus-2 surface with two disks glued in, as in Figure 3

Figure 2: The wedge sum of a cylinder and a Möbius band

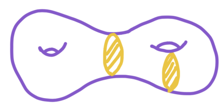


Figure 3: Two disks glued into a closed genus-2 surface

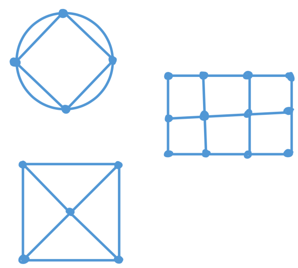


Figure 4: Three connected graphs

(d) A Klein bottle

(e) Two Möbius bands glued by the identity map along their boundaries.

(f) Each of graphs in Figure 4.

(g) The CW complex shown in Figure 5.

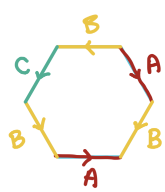


Figure 5: A 2-dimensional CW complex

(h) The plane $\mathbb{R}^2$ with n punctures

(i) The 2-sphere S^2 with n punctures

(j) $\mathbb{R}^3$ after deleting n lines through the origin

(k) The torus with n punctures

(l) $\mathbb{RP}^2$ with a puncture

(m) The quotient of the torus obtained by choosing an embedded disk D^2, and identifying its boundary S^1 to a single point

37. **(QR Exam, January 2016).** Let K be the complete graph on 4 letters, ie, K has 4 vertices, and there is a unique edge connecting each pair of distinct vertices.

(a) Calculate $\pi_1(K)$.

(b) Show that Σ_2 is not a deformation retract of any space homotopy equivalent to K.

38. Let X be the quotient space defined by the following polygon with edge identifications. Compute $\pi_1(X)$.