Name: ______________________ Score (Out of 6 points):

1. (a) (2 points) Prove that a space X is contractible if and only if the identity map id_X is homotopic to the constant map c_{x_0} at some fixed point $x_0 \in X$.

$$id_X : X \longrightarrow X \qquad\qquad c_{x_0} : X \longrightarrow X$$
$$x \longmapsto x \qquad\qquad x \longmapsto x_0$$

Solution. By definition, a space is contractible iff it is homotopy equivalent to a one-point space $\{*\}$. Suppose X is contractible. Then the (neccesarily constant) map $f : X \to \{*\}$ has some homotopy inverse $g : \{*\} \to X$. The map $g \circ f : X \to X$ is the constant map at $g(*)$, and is, by definition of homotopy equivalence, homotopic to id_X. Thus id_X is homotopic to the constant map at the point $x_0 = g(*)$.

Now suppose id_X is homotopic to a constant map c_{x_0}. Consider the constant map $f : X \to \{*\}$ and the map $g : \{*\} \to X$ that maps $*$ to x_0. Then $f \circ g = id_{\{*\}}$, and $g \circ f$ is the constant map at x_0, which is homotopic to id_X by assumption. Thus f and g are homotopy inverses, and X is contractible.

(b) (4 points) Suppose that a space X is contractible, that is, the identity map id_X is homotopic to the constant map c_{x_0} at some point $x_0 \in X$. Show that the identity map is homotopic to the constant map $c_x : X \to X$ at **any** point $x \in X$.

Solution. Fix an arbitrary point $x_1 \in X$.

Lemma. There exists a path γ in X from x_0 to x_1.

Proof of Lemma. By assumption, there is a homotopy from $H : X \times I \to X$ from id_X to the constant function at x_0. Then the restriction of H to $\{x_1\} \times I$ defines a path from x_1 to x_0: the restriction is necessarily continuous, and it takes values x_1 when $t = 0$ and x_0 when $t = 1$. To obtain the desired path from x_0 to x_1, we reverse this path: define

$$\gamma : I \longrightarrow X$$
$$t \longmapsto H(x_1, 1-t). \quad \square$$

Again let $H : X \times I \to X$ be a homotopy from the identity map id_X to the constant map at x_0, and let $\gamma : I \to X$ be a path in X from x_0 to x_1. We will interpret γ as a homotopy between the constant maps c_{x_0} and c_{x_1}, and concatenate these homotopies. Define

$$F : X \times I \longrightarrow X$$
$$(x,t) \longmapsto \begin{cases} H(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma(2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then F is continuous by the Pasting Lemma (Homework #1 Problem 1(a)) since it restricts to continuous maps on the closed subsets $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$. It gives the desired homotopy from the identity map id_X on X to the constant map at our arbitrary fixed point x_1.