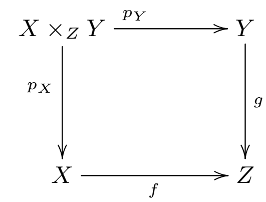
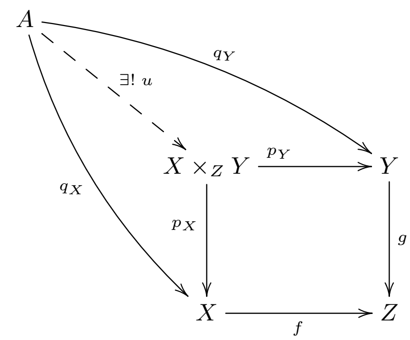
Name: ______________________________ Score (Out of 5 points):

Definition. Let $\mathcal{C}$ be a category with morphisms $f : X \to Z$ and $g : Y \to Z$. The *pullback* of f and g, if it is exists, is an object $X \times_Z Y$ of $\mathcal{C}$ along with morphisms $p_X : X \times_Z Y \to X$ and $p_Y : X \times_Z Y \to Y$ that make the following diagram commute,

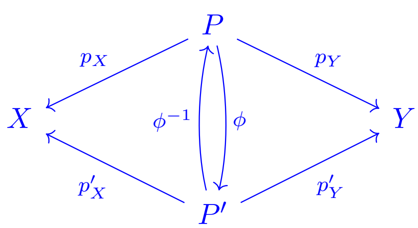


and satisfy the following universal property. Whenever there is an object A and morphisms $q_X : A \to X$ and $q_Y : A \to Y$ satisfying $f \circ q_X = g \circ q_Y$, then there is a unique morphism $u : A \to X \times_Z Y$ as shown that makes the diagram commute.



1. (a) (1 point) Let $\mathcal{C}$ be a category, and let $f : X \to Z$ and $g : Y \to Z$ be two morphisms in $\mathcal{C}$. Give a precise statement of what it means to say that the universal property defines the pullback 'uniquely up to unique isomorphism'. (You do not need to prove this statement).

Solution. Suppose that P (with maps $p_X : P \to X$ and $p_Y : P \to Y$) and P' (with maps $p'_X : P' \to X$ and $p'_Y : P' \to Y$) both satisfied the universal property. Then there exists an isomorphism $\phi : P \to P'$ in $\mathcal{C}$ that makes the following diagram commute, and it is the unique isomorphism in $\mathcal{C}$ making this diagram commute.



We saw in class that we can construct the map ϕ, show its uniqueness, and verify it is an isomorphism, by using four applications of the universal property for P and P'.

(b) (4 points) Let Top be the category of topological spaces and continuous maps, and let $f : X \to Z$ and $g : Y \to Z$ be two continuous maps. Show that the pullback $X \times_Z Y$ is the subspace of the product $X \times Y$ (with the product topology),

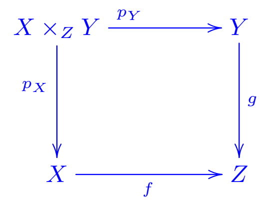
$$X \times_Z Y = \{(x, y) \mid f(x) = g(y)\} \subseteq X \times Y$$

and the maps $p_X : X \times_Z Y \to X$ and $p_Y : X \times_Z Y \to Y$ are the restrictions of the projection maps $X \times Y \to X$ and $X \times Y \to Y$, respectively.

Solution. Observe that, by construction, $X \times_Z Y$ is the subspace of $X \times Y$ that satisfies

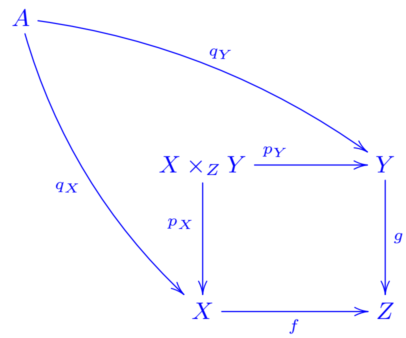
$$f \circ p_X = g \circ p_Y,$$

hence the diagram (shown to the right) commutes.



Suppose we have a commuting diagram as shown. We must prove the existence and uniqueness of a continuous map $u : A \to X \times_Z Y$ that makes the diagram commute. We define u to be the map

$$\begin{aligned} u : A &\longrightarrow X \times_Z Y \\ a &\longmapsto (q_X(a), q_Y(a)) \end{aligned}$$



We observe that, for any $a \in A$, the point $(q_X(a), q_Y(a))$ of $X \times Y$ is contained in the subspace $X \times_Z Y$, since

$$f(q_X(a)) = g(q_Y(a))$$

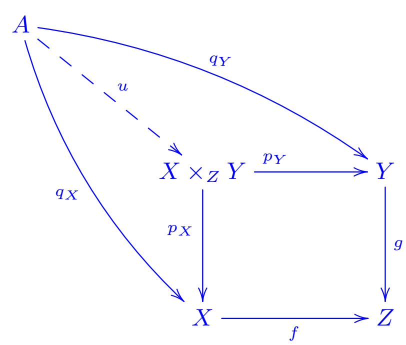
by the assumption that the diagram commutes. We observe moreover that u is continuous, since it is obtained by restricting the codomain of the continuous map

$$\begin{aligned} q_X \times q_Y : A &\longrightarrow X \times Y \\ a &\longmapsto (q_X(a), q_Y(a)) \end{aligned}$$

Finally, we observe that the adjacent diagram commutes, since

$$p_X(u(a)) = p_X\big(q_X(a), q_Y(a)\big) = q_X(a) \quad \text{for all } a \in A,$$

and similarly $p_Y(u(a)) = q_Y(a)$.



On the other hand, commutativity of the diagram also ensures uniqueness of u, since it completely determines the image of an element $a \in A$: the statement that the diagram commutes implies that the X-coordinate of $u(a)$ is $q_X(a)$ and that the Y-coordinate is $q_Y(a)$.