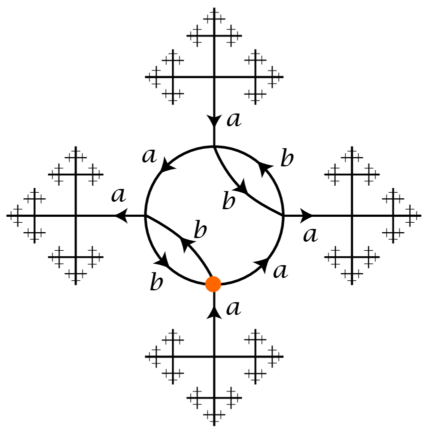
Name: ______________________________ Score (Out of 7 points):

1. (7 points) Consider the wedge $X = S^1 \vee S^1$ of circles with wedge point x_0. We use the letters a and b to label the two circles, and (by mild abuse of notation) to represent the corresponding generators of $\pi_1(X, x_0)$,which we may identify with the free group $F_{\{a,b\}}$ on a and b.

   Below is a (based) cover $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. The covering map p is specified by the edge labels and orientations. A distinguished basepoint $\widetilde{x}_0$ is marked with an orange dot.



   Let $H := p_* \left(\pi_1\left(\widetilde{X}, \widetilde{x}_0\right)\right)$.

   Fill in the following. No justification necessary, however (if applicable) please indicate your choice of maximal tree, to help me verify your solution.

   (a) The degree of this cover is ... (countably) infinite

   (b) Circle one: this cover is ... `REGULAR` `NOT REGULAR`

   (c) The deck group is (as an abstract group) ... $\mathbb{Z}/2\mathbb{Z}$

   (d) State a free generating set for H.

   One possible solution, using the maximal tree on the following page: $b^2,\ ab^{-1}ab^{-1},\ ab^2a^{-1}$

   (e) Circle precisely the elements of $F_{\{a,b\}}$ below that are contained in H.

   a a^2 b^2 ab ba ab^2 b^2ab $abab$

   (f) Circle precisely the elements of $F_{\{a,b\}}$ below that are in the normalizer $N(H)$ of H.

   a a^2 b^2 ab ba ab^2 b^2ab $abab$

Solutions, explained:

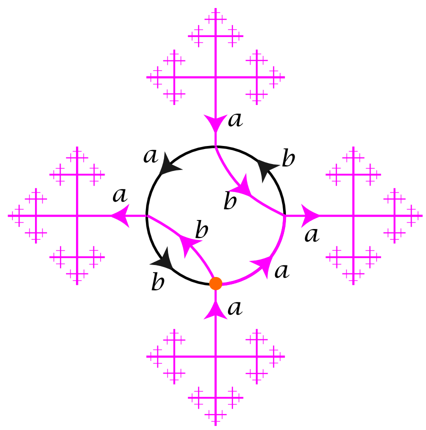
(a) Since the base space is connected, the degree is well-defined. It is the (common) cardinality of the p-preimage of any point in X. When X is the wedge $S^1 \vee S^1$, viewed as a graph with one vertex x_0, then $p^{-1}(x_0)$ is the set of vertices of the covering space, and the degree of p is the number of vertices of $\widetilde{X}$.

(b) A covering space is regular iff the group of deck transformations acts transitively on the preimage of any point in the base. In the case that X is a wedge of circles—and we encode a covering space $\widetilde{X}$ as directed labelled graph by the conventions used in this problem—a deck map is precisely a graph automorphism of $\widetilde{X}$ that respects the labels and orientations on the edges. By visual inspection, the graph on this quiz is not symmetric enough to have a transitive action on its vertices, so it is not regular.

(c) By visual inspection, the only nontrivial automorphism (of the labelled, oriented graph) is 180° rotation. Hence the deck group is the group of order 2.

(d) Steps to find a free generating set for H:

- Choose a maximal tree T in the cover, that is, a tree containing every vertex. One such tree is shown highlighted in pink.



- Since T is a contractible CW subcomplex, the quotient $\widetilde{X} \to \widetilde{X}/T$ is a homotopy equivalence. Its image $\widetilde{X}/T$ is a wedge of circles, one circle for each edge not in T.
- Choose a lift of each circle. This means, for each edge e not in T: Determine a path through $\widetilde{X}$ that starts at x_0, travels in T to a vertex of e, traverses e once, and travels in T back to x_0. (The path is unique once we specify whether it will traverse e parallel or antiparallel its orientation).
- Because the homotopy equivalence $\widetilde{X} \to \widetilde{X}/T$ induces an isomorphism of fundamental groups, these loops are a free generating set for $\pi_1(\widetilde{X}, \widetilde{x}_0)$.
- Because $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective, the images of these generators are a free generating set for $H = im(p_*)$. We record the image of each loop as a word in $F_{\{a,b\}} \cong \pi_1(X, x_0)$ by reading the labels off the edges as we traverse the loop. Specifically, if an edge is labelled by an a, we record the letter a if we traverse the edge parallel its orientation, and record the letter a^{-1} if antiparallel.

With this choice of maximal tree, and the decision to traverse the three complementary black edges in the parallel direction, we obtain the free generating set b^2, $ab^{-1}ab^{-1}$, ab^2a^{-1} for H.

(e) Each element in $\pi_1(X, x_0)$ (interpreted as a word in $F_{\{a,b\}}$) has a unique lift to a path in $\widetilde{X}$ starting at $\widetilde{x}_0$: the word prescribes the sequence of edges. Then the loop γ is in H if and only if its lift $\widetilde{\gamma}$ starting at $\widetilde{x}_0$ is a loop. (See Hatcher, Propostion 1.31.)

(f) The loop γ is in $N(H)$ if and only if its lift $\widetilde{\gamma}$ to $\widetilde{X}$ starting at $\widetilde{x}_0$ has endpoint $\widetilde{\gamma}(1)$ in the orbit of $\widetilde{x}_0$ under the deck group action. (See Hatcher, Propostion 1.37 and proof of Proposition 1.39.)