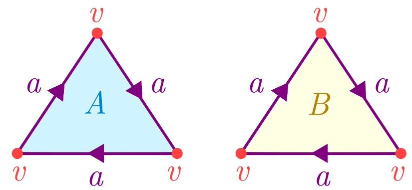
Name: ____________________ Score (Out of 4 points):

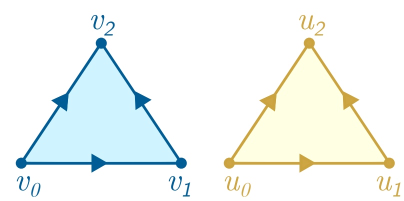
1. (4 points) A space X is formed from the following two triangles with the edge identifications shown. Compute the simplicial homology groups $H_n(X)$. Express each homology group in the sense of the structure theorem for finitely generated abelian groups.



Solution. The complex X has a single simplex in dimensions 0 and 1, and two simplices in dimension 2,

$$C_0(X) = \mathbb{Z}v \qquad C_1(X) = \mathbb{Z}a \qquad C_2(X) = \mathbb{Z}A \oplus \mathbb{Z}B \qquad C_n(X) \cong 0 \text{ for all } n \geq 3.$$

We fix the oriented simplex $[v_0, v_1, v_2]$ to be the domain of (the characteristic map of) the 2-simplex A, and fix $[u_0, u_1, u_2]$ for B. We observe that the edge $[v_0, v_1]$ is glued to $-a$, $[v_1, v_2]$ is glued to $-a$, and $[v_0, v_2]$ is glued to a. Similarly $[u_0, u_1]$ is glued to $-a$, $[u_1, u_2]$ is glued to $-a$, and $[u_0, u_2]$ is glued to a.



Thus, for example, $\partial_2(A) = A|_{[v_1,v_2]} - A|_{[v_0,v_2]} + A|_{[v_0,v_1]} = (-a) - (a) + (-a)$.

Both endpoints of the edge a are glued to v. We deduce that our boundary maps are

$$\begin{aligned} \partial_2 : C_2(X) &\longrightarrow C_1(X) \\ \mathbb{Z}A \oplus \mathbb{Z}B &\longrightarrow \mathbb{Z}a \\ A &\longmapsto (-a) - (a) - (-a) = -3a \\ B &\longmapsto (-a) - (a) - (-a) = -3a \end{aligned} \qquad \begin{aligned} \partial_1 : C_1(X) &\longrightarrow C_2(X) \\ \mathbb{Z}a &\longrightarrow \mathbb{Z}v \\ a &\longmapsto v - v = 0 \end{aligned}$$

Then our chain complex

$$\longrightarrow 0 \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \longrightarrow 0$$

is given by

$$\longrightarrow 0 \xrightarrow{\quad 0 \quad} \mathbb{Z}A \oplus \mathbb{Z}B \xrightarrow{\begin{bmatrix} -3 & -3 \end{bmatrix}} \mathbb{Z}a \xrightarrow{\quad 0 \quad} \mathbb{Z}v \longrightarrow 0.$$

$$\ker(\partial_2) = \mathbb{Z}(A - B) \qquad \ker(\partial_1) = \mathbb{Z}a \qquad \ker(\partial_0) = \mathbb{Z}v$$

$$\operatorname{im}(\partial_3) = 0 \qquad \operatorname{im}(\partial_2) = 3\mathbb{Z}a \qquad \operatorname{im}(\partial_1) = 0$$

We find

$$H_0(X) \cong \mathbb{Z} \qquad H_1(X) \cong \mathbb{Z}/3\mathbb{Z} \qquad H_2(X) \cong \mathbb{Z} \qquad H_n(X) \cong 0 \text{ for all } n \geq 3.$$