Name: ________________________________ Score (Out of 4 points):

1. (5 points) Let $Y \cong S^n$ be a smooth n-sphere, and let $X \subseteq Y$ be a smoothly embedded d-sphere, for some $0 \leq d < n$. Compute the reduced homology groups of the quotient space Y/X.

   Note: The tubular neighbourhood theorem from differential topology implies that (Y, X) is a good pair.

   This was a part of a problem on the May 2024 QR exam.

   Solution. The sphere S^p has reduced homology $\widetilde{H}_i(S^p) \cong \begin{cases} \mathbb{Z}, & i = p \\ 0, & i \neq p. \end{cases}$

   Since (Y, X) is a good pair, $\widetilde{H}_i(X/Y) \cong H(Y, X)$, and we apply the long exact sequence of a pair

$$\cdots \longrightarrow \widetilde{H}_i(S^d) \xrightarrow{\iota_*} \widetilde{H}_i(S^n) \longrightarrow \widetilde{H}_i(Y/X) \longrightarrow \widetilde{H}_{i-1}(S^d) \xrightarrow{\iota_*} \widetilde{H}_{i-1}(S^n) \longrightarrow \cdots$$

   We first consider the case that $n \neq d+1$. In this case, we obtain the short exact sequences

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & \widetilde{H}_n(S^d) & \longrightarrow & \widetilde{H}_n(S^n) & \longrightarrow & \widetilde{H}_n(Y/X) & \longrightarrow & \widetilde{H}_{n-1}(S^d) & \xrightarrow{\iota_*} & \widetilde{H}_{n-1}(S^n) & \longrightarrow \cdots \\ & \| & & \| & & & & \| & & \| & \\ & 0 & & \mathbb{Z} & & & & 0 & & 0 & \end{array}$$

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & \widetilde{H}_{d+1}(S^d) & \longrightarrow & \widetilde{H}_{d+1}(S^n) & \longrightarrow & \widetilde{H}_{d+1}(Y/X) & \longrightarrow & \widetilde{H}_d(S^d) & \xrightarrow{\iota_*} & \widetilde{H}_d(S^n) & \longrightarrow \cdots \\ & \| & & \| & & & & \| & & \| & \\ & 0 & & 0 & & & & \mathbb{Z} & & 0 & \end{array}$$

   and for all $i \neq n, d+1$,

$$\begin{array}{ccccccc} \cdots \longrightarrow & \widetilde{H}_i(S^n) & \longrightarrow & \widetilde{H}_i(Y/X) & \longrightarrow & \widetilde{H}_{i-1}(S^d) & \xrightarrow{\iota_*} \cdots \\ & \| & & & & \| & \\ & 0 & & & & 0 & \end{array}$$

   Thus if $n > d+1$, by exactness, we find $\widetilde{H}_i(Y/X) \cong \begin{cases} \mathbb{Z}, & i = n \\ \mathbb{Z}, & i = d+1 \\ 0, & \text{otherwise.} \end{cases}$

   If $n = d+1$, then the nonzero reduced homology groups of Y/X are determined by the short exact sequence

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & \widetilde{H}_n(S^d) & \longrightarrow & \widetilde{H}_n(S^n) & \xrightarrow{q_*} & \widetilde{H}_n(Y/X) & \xrightarrow{\delta} & \widetilde{H}_d(S^d) & \xrightarrow{\iota_*} & \widetilde{H}_d(S^n) & \longrightarrow \cdots \\ & \| & & \| & & & & \| & & \| & \\ & 0 & & \mathbb{Z} & & & & \mathbb{Z} & & 0 & \end{array}$$

We will use the short exact sequence to prove that $\widetilde{H}_n(Y/X) \cong \mathbb{Z}^2$.

One approach is to quote the general result from abstract algebra that, if the quotient group in a short exact sequence of abelian groups is free abelian, then the short exact sequence must split.

We also outline an argument to check this isomorphism by hand: Because $\widetilde{H}_n(Y/X)$ is an extension of finitely generated abelian groups, it is itself a finitely generated abelian group, and so is determined by its rank and torsion subgroup. The torsion subgroup must be in the kernel of the surjective homomorphism δ, since its image has no nonzero torsion elements. But then by exactness the torsion subgroup must be contained in the image of the injective map q_*, which implies it is zero. The rank-nullity theorem, applied to δ, implies that the rank of $\widetilde{H}_n(Y/X)$ is 2.

We conclude, when $n = d+1$, that

$$\widetilde{H}_i(Y/X) \cong \begin{cases} \mathbb{Z}^2, & i = n \\ 0, & \text{otherwise.} \end{cases}$$

Remark: For an alternate proof, use Hatcher Example 0.14 to argue that $Y/X \simeq S^n \vee S^{d+1}$.