Jenny Wilson 3 July 2013

An introduction to Outer Space

History

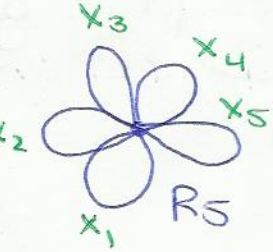
1986 - Outer space introduced by Culler-Vogtman to study (outer) automorphisms of the free gp F_n.

Properties of Outer Space X_n:

- $\mathrm{Out}(F_n) \curvearrowright X_n$ properly discontinuously (finite point stabilizers)
- X_n is contractible
- X_n decomposes as disjoint union of open simplices but is <u>not</u> a manifold, <u>not</u> cocompact

$\mathrm{Out}(F_n) \curvearrowright X_n$ analogous to $\mathrm{Mod}(S) \curvearrowright \mathrm{Teich}(S)$.

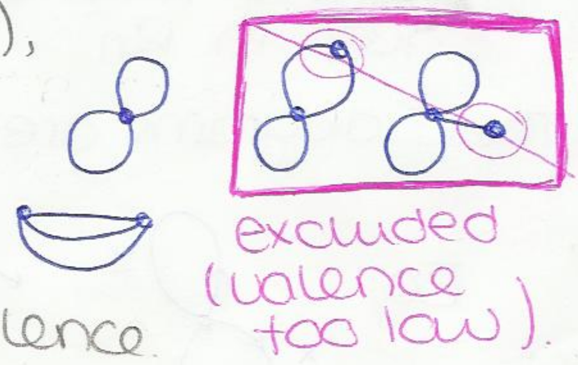
Definition



Let R_n be a rose (1 vertex, n edges)

Fix identification $\pi_1(R_n) \cong F_n = \langle x_1, \dots, x_n \rangle$ each x_i to an edge.

<u>Def^n</u> <u>Outer space</u> X_n is a space of equivalence classes of marked metric graphs (Γ, g, ℓ),



- Γ is a graph, $\pi_1(\Gamma) = F_n$, all vertices valence ≥ 3.
- $g : R_n \longrightarrow \Gamma$ the <u>marking</u>, an (unbased) homotopy equivalence

The marking identifies $\pi_1(\Gamma)$ with F_n.

- ℓ is a metric $\ell : \{\text{edges of } \Gamma\} \longrightarrow (0,\infty)$

assigns each edge a positive real length.

We will normalize so that

$$\mathrm{vol}(\ell) := \sum_{\text{edges } e} \ell(e) = 1$$

Equivalence relation: $(\Gamma, f, \ell) \sim (\Gamma', f', \ell')$ if $\exists$ isometry ϕ

$$\begin{tikzcd} & \Gamma \arrow[dd, "\phi"] \\ R_n \arrow[ur, "f"] \arrow[dr, "f'"'] \arrow[r, phantom, "\circlearrowright"] & \\ & \Gamma' \end{tikzcd}$$

isometry

Diagram commutes up to homotopy.

<u>Inverse Markings</u>

A marked graph (f, Γ) can be conveniently represented by an inverse marking

$$\begin{tikzcd} R_n \arrow[r, bend left, "f"] & \Gamma \arrow[l, bend left, "g"] \end{tikzcd}$$

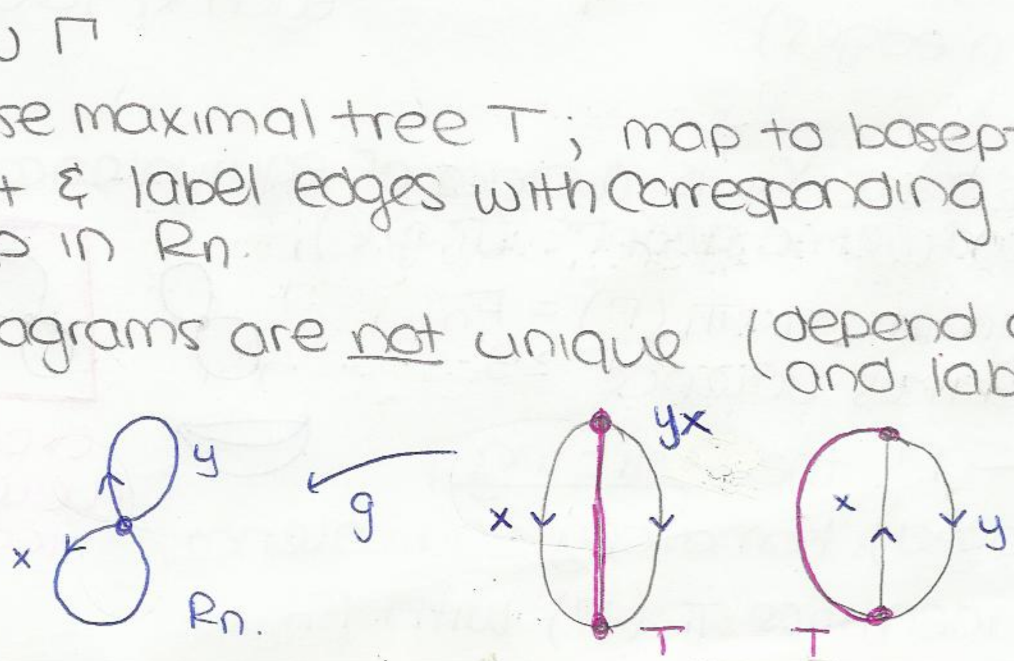
g a homotopy inverse to f.

depicted as follows:

- Draw Γ
- Choose maximal tree T; map to basept
- Orient & label edges with corresponding edge-path loop in R_n

These diagrams are <u>not</u> unique (depend on T and labelling)

Eg.



equivalent markings.

- Lengths of loops

Let $\mathcal{C}$ be set of conj. classes of F_n (each represented by a cyclically reduced word)

$\exists$ map $X_n \longrightarrow \mathbb{R}^{\mathcal{C}}$

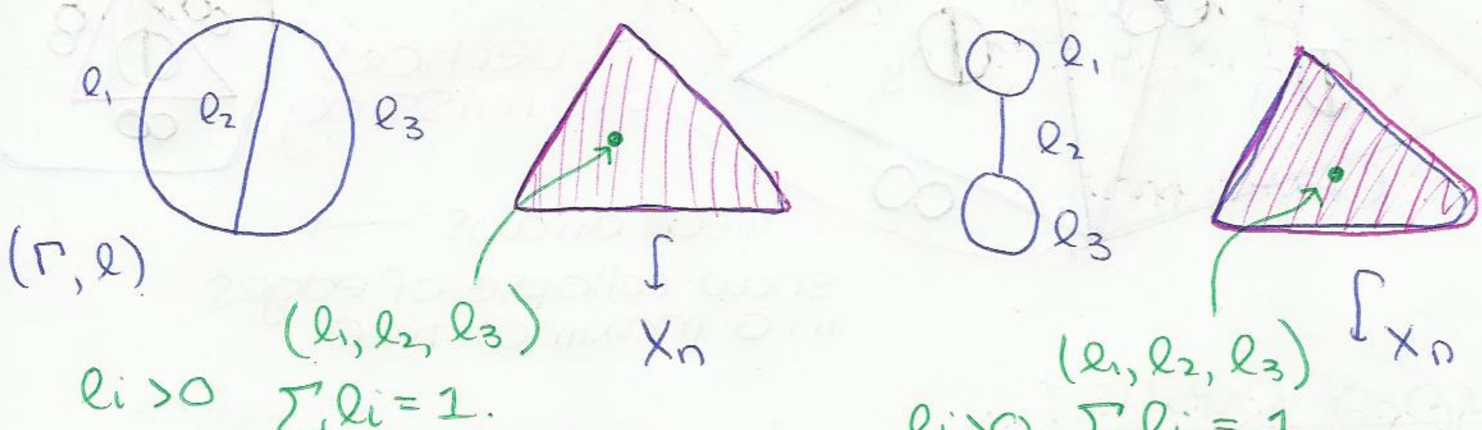
$$(\Gamma, g, \ell) \longmapsto \left\{ \begin{array}{l} \mathcal{C} \longrightarrow \mathbb{R} \\ \alpha \longmapsto \ell_\Gamma(\alpha) \end{array} \right\}$$

where $\ell_\Gamma(\alpha)$ = length of unique immersed representative of loop homotopic to $g(\alpha)$

These length functions induce a topology on X_n.

Simplices (another means of understanding the topology)

X_n is simplicial complex with missing faces.

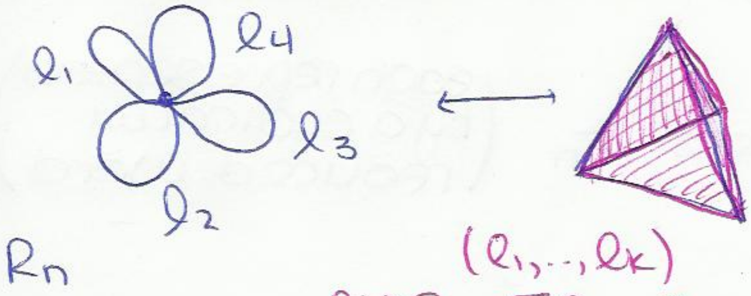


each marking of Γ corresponds to a simplex in X_n.

varying $\ell = (\ell_1, \dots, \ell_k)$ traces out a simplex.

However, there are faces missing, since we can only allow some (and not all) side lengths ℓ_i tend to 0 without changing homotopy type of Γ.

Sides can only be collapsed if they are part of a tree in Γ – for example, all loops must keep positive length.

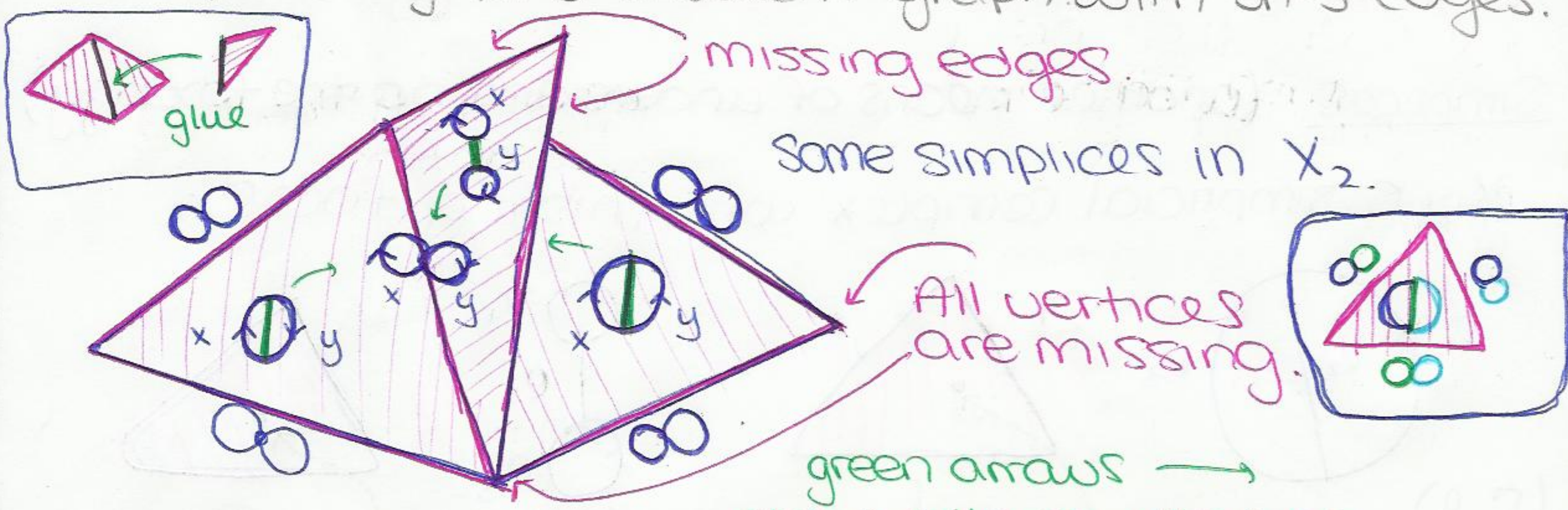


The smallest cells are dimension $(n-1)$;
Γ cannot have fewer edges than a rose R_n.

$(\ell_1, \dots, \ell_k)$
$\ell_i > 0$, $\Sigma \ell_i = 1$,

minimal dimension simplices.

Exercise: (Using Euler characteristic) show the maximal cells are dimension $3n-4$, corresponding to a trivalent graph with $3n-3$ edges.



Some simplices in X_2.

green arrows $\longrightarrow$
show collapse of edges in a maximal tree.

Action of Out(F_n)

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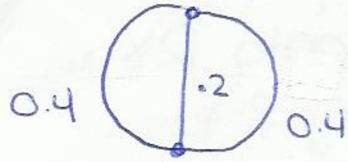
- Right action of Out(F_n) on X_n by precomposition

$\Phi \in \text{Out}(F_n)$ gives homotopy equiv $R_n \to R_n$.

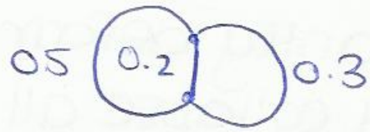
Define: $(\Gamma, g, \ell) \cdot \Phi = (\Gamma, g \circ \Phi, \ell)$

$$\Phi \circlearrowright R_n \xrightarrow[g]{} \Gamma$$

- Action is proper.
- Action is simplicial.
- $\ell_\Gamma(\Phi(\alpha)) = \ell_{\Gamma\Phi}(\alpha)$
- Stabilizer of $(\Gamma, g, \ell) \cong \text{Isom}(\Gamma)$ is finite



Stabilizer $(\Gamma, g, \ell) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.



Trivial Stabilizer.

Stabilizer $(\Gamma, f, \ell) \ni \Phi$

$$\begin{tikzcd} R_n \arrow[r, "f"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\circlearrowleft"] & \Gamma \arrow[d, "\phi"] \\ R_n \arrow[r, "f"] & \Gamma \end{tikzcd}$$

ϕ isometry

Variants: Metric Trees

For each marked metric graph (Γ, g, ℓ), universal cover $\tilde{\Gamma}$ is a metric tree with free F_n action

so we get dictionary

metric graph (Γ, ℓ) ⟷	metric tree $(\tilde{\Gamma}, \tilde{\ell})$
marking g ⟷	F_n action on $\tilde{\Gamma}$
(Equivalence $(\Gamma, g, \ell) \sim (\Gamma', g', \ell')$) ⟷	(equivariant isometry $\tilde{\Gamma} \to \tilde{\Gamma}'$)
$\mathrm{vol}(\ell) = 1$ ⟷	covolume $\tilde{\Gamma} \curvearrowleft F_n$ is 1
Length $\ell_\Gamma(\alpha)$ ⟷	translation length for α

so $X_n = \{(\Gamma, g, \ell)\}/\sim$ can be redefined as

$$X_n = \left\{ \begin{matrix} \text{minimal metric simplicial} \\ \text{free } F_n\text{-trees, covolume } 1 \end{matrix} \right\} \Big/ \begin{matrix} \text{equivar.} \\ \text{isometry} \end{matrix}$$

Subcomplexes: Reduced Outer Space RX_n

Def^n Edge e of Γ is separating if $\Gamma - e$ is disconnected

Def^n $RX_n = \left\{ (\Gamma, g, \ell) \,\middle|\, \begin{matrix} \Gamma \text{ has no} \\ \text{separating edges} \end{matrix} \right\} /\sim \;\subseteq X_n$

RX_n is reduced outer space

X_n equivariantly deformation retracts onto RX_n
(uniformly collapse all separating edges.)

Subcomplexes: The spine K_n.

- Spine $K_n \subseteq X_n$ $(2n-3)$-dimensional simplicial cpx.
  - an equivariant deformation retract of X_n (hence contractible)
  - cocompact
  - finite stabilizers

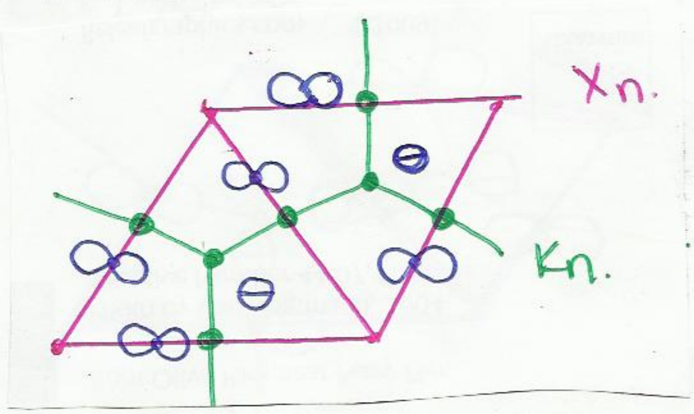
so $K_n \simeq Out(F_n)$ quasi-isometry

<u>Thm</u> (Bridson-Vogtmann, 2001)

$Out(F_n) \subseteq$ simplicial aut. group of K_n.

this is an equality.

<u>Defn</u> $K_n \subseteq X_n$ can be defined via barycentric subdivision



K_n has a vertex for each simplex

we can equivalently define K_n as a simplicial complex as follows:

- vertices are marked graphs (Γ, g) (no metric) with equivalence relation

$$\begin{tikzcd} & \Gamma \arrow[dd, "\phi"] \\ R_n \arrow[ur, "g"] \arrow[dr, "g'"'] & \\ & \Gamma' \end{tikzcd}$$

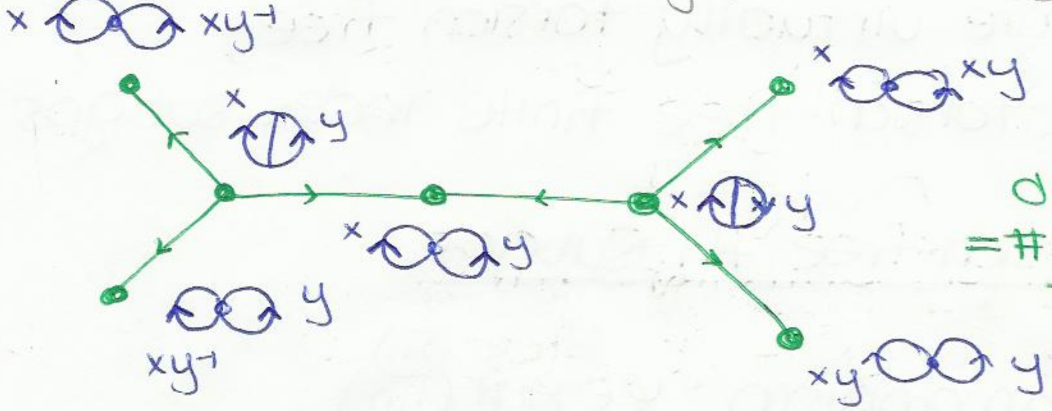
ϕ homeomorphism

$(\Gamma, g) \sim (\Gamma', g')$ if diagram commutes up to homotopy.

- k-simplex $\{(g_0, \Gamma_0), \dots, (g_k, \Gamma_k)\}$

whenever (g_i, Γ_i) can be obtained from (g_{i-1}, Γ_{i-1}) by collapsing a forest in Γ_{i-1}.

└ disjoint union of trees.



dim $K_n = 2n-3$
= # edges in max. tree

ie,

K_n = geometric realization of poset of open simplices in X_n.

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X_n is contractible

Thm (Cullen-Vogtman, 1986) Outerspace is contractible.

Pf Induction on (enumerated) roses in X_n;
combinatorial Morse theory

Finiteness Properties

<u>Thm</u> (Nielsen 1924, McCool 1974)
$Aut(F_n)$ and $Out(F_n)$ are finitely presented.

<u>NB</u> $Aut(F_n)$ and $Out(F_n)$ have torsion elements, so they cannot have finite cohomological dim.

However, they are virtually torsion free, ie, they have torsion-free finite index subgps.

Constructing torsion-free f.i. subgps

Given an automorphism $\varphi \in Aut(F_n)$, and the abelianization $F_n \longrightarrow \mathbb{Z}^n$

The composite $F_n \xrightarrow{\varphi} F_n \longrightarrow \mathbb{Z}^n$

has abelian codomain, and so factors through the abelianization of the domain

$$\begin{tikzcd} F_n \arrow[r, "\varphi"] \arrow[d] & F_n \arrow[r] & \mathbb{Z}^n \\ \mathbb{Z}^n \arrow[urr, dashed, "\tilde{\varphi}"] & & \end{tikzcd}$$

giving a map $\begin{aligned} Aut(F_n) &\longrightarrow Aut(\mathbb{Z}^n) = GL_n(\mathbb{Z}) \\ \varphi &\longmapsto \tilde{\varphi} \end{aligned}$

Inner automorphisms are in the kernel,

so we get a map

The map surjects.

$$\mathrm{Out}(F_n) \longrightarrow GL_n(\mathbb{Z}).$$

<u>Thm</u> (Baumslag–Taylor 1968)

The kernels are torsion-free:

$$1 \to JA_n \longrightarrow \mathrm{Aut}(F_n) \longrightarrow GL_n(\mathbb{Z}) \longrightarrow 1$$

<u>So</u> we can construct torsion-free finite index subgroups of $\mathrm{Aut}(F_n)$ or $\mathrm{Out}(F_n)$ by pulling back torsion-free finite index subgps of $GL_n(\mathbb{Z})$.

<u>$\mathrm{Out}(F_n)$ is WFL</u>

<u>Def^n</u> A group G is type FL if $\mathbb{Z}$ admits a finite free resolution over $\mathbb{Z}G$.

ie, a free resolution which is

- finite in length
- finite type (each group finitely generated)

FL F for finite, L for free ("libre")

this condition is much stronger than finite cohomological dimension.

<u>Defn</u> A group G is WFL if

- G is virtually torsion free
- every torsion-free f.i. subgroup is type FL.

<u>Eg</u> Arithmetic groups, Mapping Class Groups are WFL.

<u>Thm</u> $Out(F_n)$ is WFL.

<u>Pf</u> The spine K_n of outer space has a free properly discontinuous action by any torsion-free f.i. subgp $\Gamma \leq Out(F_n)$

(since stabilizers were torsion groups).

so K_n / Γ is a compact $(2n-3)$-dim $K(\Gamma, 1)$

and the chain complex for the (contractible) space K_n is a finite free resolution for $\mathbb{Z}$ over $\mathbb{Z}\Gamma$

Thm The UCD of $Out(F_n)$ is $2n-3$
The UCD of $Aut(F_n)$ is $2n-2$.

Pf (For $Out(F_n)$)
- Upper bound: $\dim(K_n) = 2n-3$
- Lower bound: $Out(F_n) \geq \mathbb{Z}^{2n-3}$,

image of automorphisms:

$$\rho_i : \begin{cases} x_i \longmapsto x_i x_1 \\ x_j \longmapsto x_j \end{cases} j \neq i \qquad \lambda_i : \begin{cases} x_i \longmapsto x_1 x_i \\ x_j \longmapsto x_j \end{cases} j \neq i.$$

so $\operatorname{ucd}(Out(F_n)) \geq \operatorname{cd}(\mathbb{Z}^{2n-3}) = 2n-3.$

Residual Finiteness

Defn A group G is <u>residually finite</u> if for every $g \neq 1$ in G, $\exists$ homomorphism $h : G \longrightarrow H$ with H finite, $h(g) \neq 1$.

"Elements of G are detectable by finite quotients"

Thm (Baumslag 1963, Grossman 1974)
$Aut(F_n)$ and $Out(F_n)$ are residually finite.

Homological Stability & Homolog Computations.

Thm (Hatcher 1995, Hatcher-Vogtmann 1996).
2004

$Aut(F_n) \longrightarrow Aut(F_{n+1})$ induces

$$H_i(Aut(F_n)) \xrightarrow{\cong} H_i(Aut(F_{n+1}))$$

for $n \geq 2i+2$

$Aut(F_n) \twoheadrightarrow Out(F_n)$ induces

$$H_i(Aut(F_n)) \xrightarrow{\cong} H_i(Out(F_n))$$

for $n \geq 2i+4$

So both $Out(F_n)$ and $Aut(F_n)$ are homologically stable over $\mathbb{Z}$ with linear range.

Pf Idea Hatcher-Vogtmann study a space of 3-manifolds,

doubled handlebody

$$M_{n,s} = \#_n S^1 \times S^2 - \left\{ \begin{array}{c} s \text{ disjoint} \\ 2\text{-spheres} \end{array} \right\}$$

and identify certain quotients of the mapping class groups with $Aut(F_n)$ and $Out(F_n)$.

Outerspace can be identified as a certain subspace of the parameter space of $M_{n,s}$.

The proof is analogous to Harer stability for mapping class groups of surfaces.

<u>Thm</u> (Galatius 2011)

$$\mathbb{Z} \times BAut(F_\infty)^+ \simeq \Omega^\infty S^\infty$$

↑ homotopy equivalence

so $H_*(Aut(F_n))$ is (stably) the homology of an infinite loop space.

In fact, let Σ_n := symmetric gp on n letters.

$$\Sigma_n \hookrightarrow Aut(F_n)$$ automorphisms permuting generators.

This map induces

$$H_k(\Sigma_n) \xrightarrow{\simeq} H_k(Aut(F_n)) \quad \text{for } n > 2k+1.$$

<u>Cor</u> The stable rational homology of $Out(F_n)$ vanishes.

<u>Pf</u> uses a variation on outer space involving "noncompact graphs"