Stability in the Homology of Configuration Spaces

Jenny Wilson (Stanford)
joint with Jeremy Miller (Purdue)

No Boundaries: Groups in Algebra, Geometry, and Topology
27–29 October 2017

Configuration spaces

Definition (configuration space)

M – connected non-compact finite-type manifold of dim ≥ 2

$F_k(M)$ – (ordered) configuration space of M on k points

$$F_k(M) := \{(m_1, m_2, \ldots, m_k) \in M^k \mid m_i \neq m_j \text{ for all } i \neq j\}$$

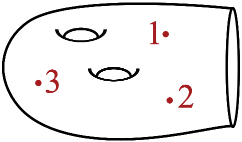

Figure: A point in $F_3(M)$

Goal: Understand $H_*(F_k(M))$

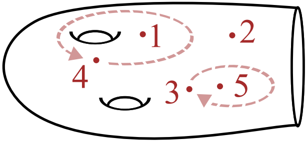

Figure: A class in $H_2(F_5(M))$

$$S_k \curvearrowright F_k(M)$$

Representation Stability

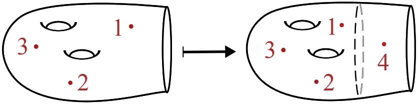

Figure: Stabilization Map $t : F_k(M) \to F_{k+1}(M)$

Strategy: Fix M. Package the sequence $\{H_*(F_k(M))\}_k$ into a module over a category encoding S_k–actions and embeddings.

Theorem (Church–Ellenberg–Farb, M–W (non-orientable case))

For each fixed i, $\{H_i(F_k(M))\}_k$ is **representation stable**.

$$\mathbb{Z}[S_{k+1}] \cdot t_*(H_i(F_k(M);\mathbb{Z})) = H_i(F_{k+1}(M);\mathbb{Z}) \qquad \textit{for } k \geq 2i.$$

Higher-Order Representation Stability

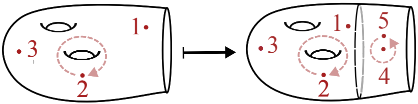

Figure: Secondary stabilization map $t' : H_i(F_k(M)) \to H_{i+1}(F_{k+2}(M))$

Theorem (M–W)

$\{H_*(F_k(M);\mathbb{Q})\}_k$ *has* **secondary representation stability***.*

For each fixed i, the sequence of "unstable" homology in

$$\left\{H_{\frac{k+i}{2}}(F_k(M);\mathbb{Q})\right\}_k$$

is finitely generated under the actions of maps t' and the groups S_k.