Representation stability and the braid groups

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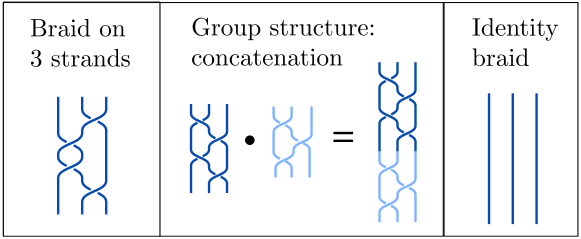
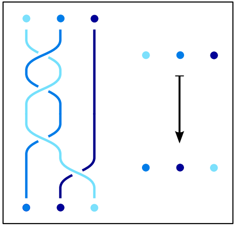
The (pure) braid group and its generalizations

Definition (braid groups)

The *braid group* B_n is the group of equivalence classes of braid diagrams under concatenation.



$\psi : B_n \longrightarrow S_n$

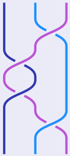


Definition (pure braid groups)

The *pure* braid group

$$P_n := \ker(\psi)$$

is the subgroup of *pure* braids.

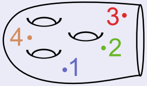


Configuration spaces

Definition (Ordered configuration space)

M – topological space

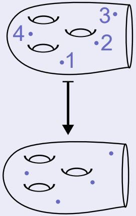
$F_n(M)$ – (ordered) configuration space of M on n points

Eg,  $\in F_4(M)$

$$F_n(M) := \{(m_1, m_2, \ldots, m_n) \in M^n \mid m_i \neq m_j \text{ for all } i \neq j\} \subseteq M^n$$

Definition (Unordered configuration space)

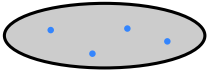
$$F_n(M)$$
$$\downarrow$$
$$C_n(M) := F_n(M)/S_n$$



$C_n(M)$ – unordered configuration space of M on n points

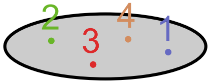
$$C_n(M) \cong \left\{ \begin{array}{c} n\text{-element} \\ \text{subsets of } M \end{array} \right\}$$

$B_n \cong \pi_1(C_n(\mathbb{C}))$



$\in C_4(\mathbb{C})$

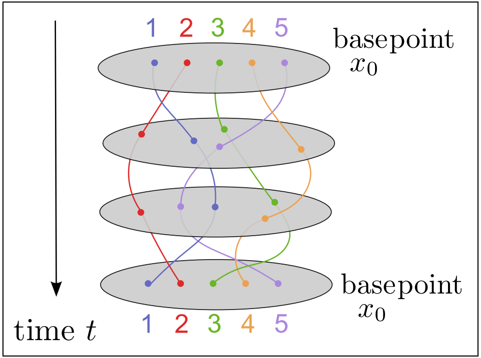
$P_n \cong \pi_1(F_n(\mathbb{C}))$



$\in F_4(\mathbb{C})$

These spaces are $K(\pi, 1)$'s!

Loop $\gamma(t)$ in $F_n(\mathbb{C})$:



Generalization: $C_n(M)$ and $F_n(M)$, for M an open connected smooth manifold, dim $\geqslant 2$

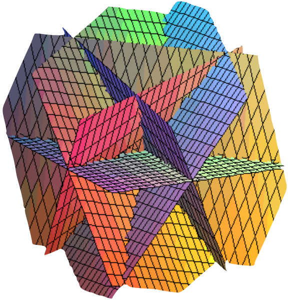
(2) The pure braid group as π_1 of a hyperplane complement

$F_n(\mathbb{C}) \cong$ complex hyperplane complement

$$\mathbb{C}^n \setminus \left\{ \begin{array}{c} \text{hyperplanes} \\ x_i = x_j, \\ 1 \leqslant i < j \leqslant n \end{array} \right\}$$

$C_n(\mathbb{C}) \cong$ complex hyperplane complement $/S_n$

A_3 real hyperplane arrangement



Graphic by John Stembridge

Generalization: Other families of complements of hyperplane arrangements in $\mathbb{C}^n$, e.g., corresponding to families of finite reflection groups

(3) The (pure) braid group as a (pure) motion group

$$F_n(\mathbb{C}) \cong \left\{ \begin{array}{c} \text{embeddings} \\ \{1, 2, 3, \ldots, n\} \hookrightarrow \mathbb{C} \end{array} \right\}$$

$$C_n(\mathbb{C}) \cong F_n(\mathbb{C})/S_n$$

Generalization: *(pure) motion groups*

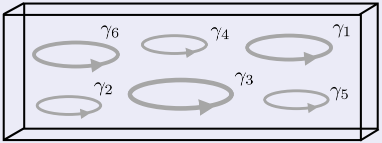
P – connected smooth manifold

Equivalence classes of paths in a space of smooth embeddings $\bigsqcup_n P \hookrightarrow \mathbb{R}^N$

Motion groups

E.g. *string motion group* Σ_n

$$C_n = \text{smooth embedding} \coprod_n S^1 \hookrightarrow \mathbb{R}^3$$

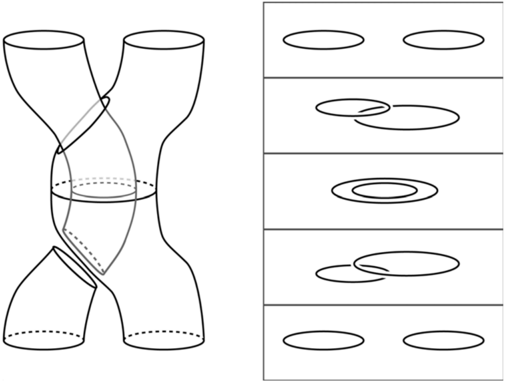


(unknotted, unlinked)

string motion – (equivalence class of) paths f_t of diffeos of $\mathbb{R}^3$,

- $f_0 = id_{\mathbb{R}^3}$
- f_1 stabilizes C_n

String motion:



Graphic by Baez–Crans–Wise

The (pure) mapping class group

Definition ((pure) mapping class group)

M_n – connected smooth manifold with n marked points (or punctures)

$\mathrm{Mod}^n(M_n)$ – mapping class group of M_n

$\mathrm{Mod}^n(M_n) :=$ (orientation-preserving) diffeos of M fixing ∂M & stabilizing the set of marked points, up to smooth isotopy fixing ∂M & the marked points

$P\mathrm{Mod}^n(M_n)$ – *pure* mapping class group of M_n

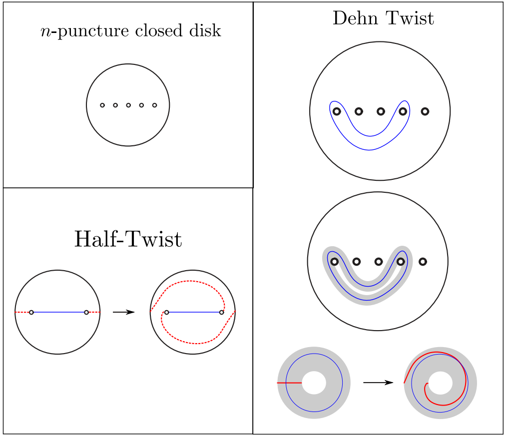
$P\mathrm{Mod}(M_n) :=$ (orientation-preserving) diffeos of M fixing ∂M & **fixing** the marked points, up to smooth isotopy fixing ∂M & the marked points

(4) (pure) braid group as a (pure) mapping class group

D_n^2 – closed 2-disk with n punctures

$B_n \cong \mathrm{Mod}^n(D_n^2)$

$P_n \cong P\mathrm{Mod}^n(D_n^2)$



Generalization: $\mathrm{Mod}^n(M_n)$ and $P\mathrm{Mod}^n(M_n)$, for M_n a smooth manifold with $\partial M \neq \varnothing$ and n distinguished punctures

Homological stability for B_n

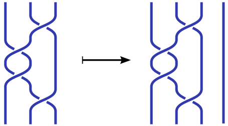
Definition (Homological stability)

$0 \overset{\phi_0}{\hookrightarrow} G_1 \overset{\phi_1}{\hookrightarrow} G_2 \overset{\phi_2}{\hookrightarrow} \dots$ – family of groups or spaces

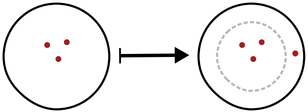
The family $\{G_n, \phi_n\}_n$ is *homologically stable* if, for each fixed degree $k \geqslant 0$,

$$(\phi_n)_* : H_k(G_n) \longrightarrow H_k(G_{n+1}) \qquad \text{is an iso } \forall n \gg k.$$

$\phi_n : B_n \longrightarrow B_{n+1}$



$\phi_n : C_n(\mathbb{C}) \longrightarrow C_{n+1}(\mathbb{C})$



Theorem (Arnold 70s)

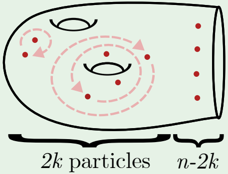
The braid groups $\{B_n, \phi_n\}$ are homologically stable.

Homologically stable families

The following families are homologically stable:

- Symmetric groups S_n (Nakoaka 1960s)
- Configuration spaces $C_n(M)$, M an open connected manifold of dim $\geqslant 2$ (McDuff 1970s)
- Certain orbit spaces of complex hyperplane complements (Brieskorn 1970s, . . .)
- General linear groups $\mathrm{GL}_n(R)$ for certain R (Quillen, Maazen, van der Kallen, Charney . . .)
- Mapping class groups of genus-n surfaces (Harer 1980s)
- Automorphisms of the free group $\mathrm{Aut}(F_n)$ (Hatcher–Vogtmann 1990s)
- Mapping class group $\mathrm{Mod}^n(M)$, for M connected, dim $\geqslant 2$, n punctures (Hatcher–Wahl 2010s)
- String motion groups Σ_n (Hatcher–Wahl 2010s)
- . . . many more . . .

Classes in $H_k(C_n(M))$:

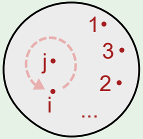


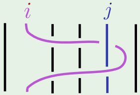
Homological stablility for P_n?

Question: Is $\{P_n, \phi_n\}_n$ homologically stable? **Answer:** No!

Eg, $H_1(P_n) = \mathbb{Z}^{\binom{n}{2}}$

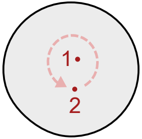
rank $\sim n^2$

generators $\alpha_{i,j}$ 

$\in H_1(F_n(\mathbb{C}))$, 

$\in P_n$

Up to action of S_n
and stabilization map ϕ_n,
$\{H_1(F_n(\mathbb{C}))\}_n$ is generated by:

$\alpha_{1,2} =$ 

$\in H_1(F_2(\mathbb{C}))$

Representation stability for P_n

Church–Farb: Fix k. The decomposition of $H_k(P_n;\mathbb{Q})$ into irreducible S_n–reps stabilizes for $n \geqslant 4k$.

$$
\begin{aligned}
Eg, H_1(P_2;\mathbb{Q}) &\cong V_{(2)} \\
H_1(P_3;\mathbb{Q}) &\cong V_{(3)} \oplus V_{(2,1)} \\
H_1(P_4;\mathbb{Q}) &\cong V_{(4)} \oplus V_{(3,1)} \oplus V_{(2,2)} \\
H_1(P_5;\mathbb{Q}) &\cong V_{(5)} \oplus V_{(4,1)} \oplus V_{(3,2)} \\
H_1(P_6;\mathbb{Q}) &\cong V_{(6)} \oplus V_{(5,1)} \oplus V_{(4,2)} \\
H_1(P_7;\mathbb{Q}) &\cong V_{(7)} \oplus V_{(6,1)} \oplus V_{(5,2)}
\end{aligned}
$$

Question: What underlying structure is driving these patterns?

A theorem of Church–Ellenberg–Farb

Answer: They are finitely generated FI♯–modules.

Theorem (Church–Ellenberg–Farb 2010s)

Family $\{G_n\}_n$. Suppose for each fixed k, $\{H_k(G_n)\}_n$ is a finitely generated FI♯*-module in degree $\leqslant d_k$. Then ...*

- **finite generation**

$$\mathbb{Z}[S_{n+1}] \cdot (\phi_n)_*(H_k(G_n)) = H_k(G_{n+1}) \qquad \textit{for } n \geqslant d_k.$$

- **polynomial Betti numbers**

$$\dim_{\mathbb{Q}} H_k(G_n; \mathbb{Q}) = \textit{a polynomial in } n \textit{ of degree} \leqslant d_k$$

Eg, $\dim_{\mathbb{Q}} H_1(P_n; \mathbb{Q})$

$$= \binom{n}{2} = \tfrac{1}{2}(n)(n-1)$$

A theorem of Church–Ellenberg–Farb

Theorem (ctd.)

- **multiplicity stability**

  The decomposition of $H_k(G_n;\mathbb{Q})$ into irreducible S_n–reps stabilizes for $n \geqslant 2d_k$.

- **character polynomials**

  The character $\chi_{H_k(G_n;\mathbb{Q})}$ is a polynomial in the "cycle-counting" functions, independent of n.

- **free module structure**

  $H_k(G_n)$ is an induced module of a certain form, induced from certain specific subreps of

  $$H_k(G_0), H_k(G_1), \ldots, H_k(G_{d_k})$$

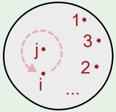
Eg, $\chi_{H_1(P_n;\mathbb{Q})}(\sigma)$

$$= (\#\text{2-cycles in } \sigma) + \binom{\#\text{1-cycles in } \sigma}{2}$$

for $\sigma \in S_n$, for all n.

Eg, $H_1(P_n) = \bigoplus_{\{i,j\}\subseteq\{1,2,\ldots,n\}} \mathbb{Z}\alpha_{i,j}$

$$\cong \mathrm{Ind}_{S_2 \times S_{n-2}}^{S_n} H_1(P_2)$$

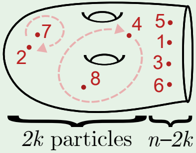
$\alpha_{i,j} =$ 

Other instances of representation stability

The (co)homology groups are finitely generated FI♯-modules for ...

- Ordered configuration spaces $F_n(M)$, M an open connected manifold of dim $\geqslant 2$ (Church, Church–Ellenberg–Farb, Miller–W 2010s)
- Certain complex hyperplane complements (Church–Ellenberg–Farb, W 2010s)
- Pure string motion group $P\Sigma_n$ (W 2010s)
- $P\text{Mod}^n(M)$, M connected smooth manifold, $\dim M \geqslant 2$, $\partial M \neq 0$ (Jimenez Rolland, 2010s)
- Pure virtual braid groups PvB_n & pure flat braid groups PfB_n (Lee, 2010s)
- ... many more ...

$H_k(F_n(M))$ is spanned by:



FI-modules and FI♯-modules

Definition (FI and FI-modules)

Let FI denote the category of **F**inite sets and **I**njective maps.

An FI-*module* is a functor V : FI → Ab Gps.

$$\begin{array}{ccccccccccc} \{1\} & \hookrightarrow & \{1,2\} & \hookrightarrow & \{1,2,3\} & \hookrightarrow & \{1,2,3,4\} & \hookrightarrow & \{1,2,3,4,5\} & \hookrightarrow & \\ \circlearrowright & & \circlearrowright & & \circlearrowright & & \circlearrowright & & \circlearrowright & & \\ S_1 & & S_2 & & S_3 & & S_4 & & S_5 & & \\ & & & & & \Big\downarrow V & & & & & \\ V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & V_4 & \longrightarrow & V_5 & \longrightarrow & \\ \circlearrowright & & \circlearrowright & & \circlearrowright & & \circlearrowright & & \circlearrowright & & \\ S_1 & & S_2 & & S_3 & & S_4 & & S_5 & & \end{array}$$

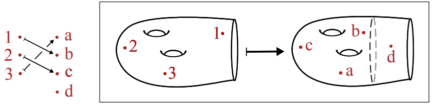
V is an FI♯-*module* if it admits is both a covariant and contravariant functor FI → Ab Gps (in compatible way).

Some FI♯-modules

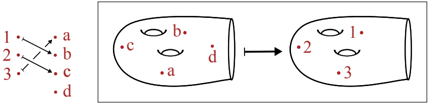
Examples of FI♯-modules

Example:	$\mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \cdots$	trivial S_n-reps
Example:	$\mathbb{Z} \hookrightarrow \mathbb{Z}^2 \hookrightarrow \mathbb{Z}^3 \hookrightarrow \cdots$	canonical S_n permutation reps
Example:	$\mathbb{Z}[x_1] \hookrightarrow \mathbb{Z}[x_1, x_2] \hookrightarrow \mathbb{Z}[x_1, x_2, x_3] \hookrightarrow \cdots$	S_n permutes variables
Non-Example:	$\mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \cdots$	alternating S_n-reps
Non-Example:	$\mathbb{Z}[S_1] \hookrightarrow \mathbb{Z}[S_2] \hookrightarrow \mathbb{Z}[S_3] \hookrightarrow \cdots$	left regular S_n-reps
Example:	$H_k(F_1(M)) \to H_k(F_2(M)) \to H_k(F_3(M)) \to \ldots$	(M open manifold, dim $\geqslant 2$)

FI-action:



FI^{op}-action:



Finite generation

Finite generation

Homogeneous degree-2 polynomials in $\mathbb{Z}[x_1, x_2, \ldots, x_n]$.

$$\begin{array}{ccccccc}
S_1 & & S_2 & & S_3 & & \\
\circlearrowright & & \circlearrowright & & \circlearrowright & & \\
R[x_1]_{(2)} & \hookrightarrow & R[x_1, x_2]_{(2)} & \hookrightarrow & R[x_1, x_2, x_3]_{(2)} & \hookrightarrow & \\
\parallel & & \parallel & & \parallel & & \\
\langle x_1^2 \rangle & & \langle x_1^2, x_2^2, x_1 x_2 \rangle & & \langle x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_3x_2 \rangle & & \\
\Cup & & \Cup & & & & \\
x_1^2 & & x_1 x_2 & & & & \\
\end{array}$$

Generators

$\{\mathbb{Z}[x_1, \ldots, x_n]_{(2)}\}_n$ is *finitely generated in degree* $\leqslant 2$ by generators

$$x_1^2 \in V_1, \quad x_1 x_2 \in V_2.$$

Current directions

Goals:

- Develop commutative algebra tools for proving finiteness properties of FI- or FI$\sharp$-modules.
- Adapt tools to study other categories (eg) encoding actions of groups other than S_n.

Thank you!