Representation Stability

Jenny Wilson (Michigan)

University of Chicago
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Motivating example: configuration spaces

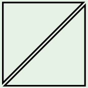
Definition (configuration space)

M – topological space

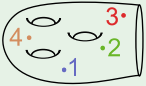
$F_n(M)$ – (ordered) configuration space of M on n points

$$F_n(M) := \{(m_1, m_2, \ldots, m_n) \in M^n \mid m_i \neq m_j \text{ for all } i \neq j\} \subseteq M^n$$

$F_n(M) = M^n \setminus$ “fat diagonal”

Eg, $F_2([0,1]) =$ 

$$F_n(M) = \left\{ \begin{array}{c} \text{embeddings} \\ \{1, 2, 3, \ldots, n\} \hookrightarrow M \end{array} \right\}$$

Eg, 

$\in F_4(\Sigma)$

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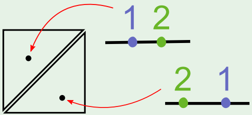
Eg, $F_2([0,1]) =$ 

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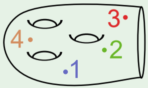
Eg, 

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Motivating example: configuration spaces

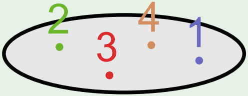
Definition (configuration space)

$$F_n(M) := \{(m_1, m_2, \ldots, m_n) \in M^n \mid m_i \neq m_j \text{ for all } i \neq j\} \subseteq M^n$$

$\left\{ \begin{array}{c} \text{Connected components} \\ \text{of } F_n([0,1]) \end{array} \right\} \longleftrightarrow S_n$

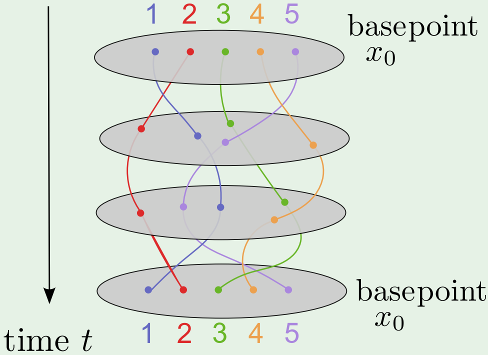
$\in F_4([0,1])$

$F_n(D^2)$ is connected



$\in F_4(D^2)$

$\pi_1(F_n(D^2)) =$ Artin's pure braid group PB_n

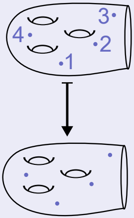


Unordered configuration spaces

$S_n \circlearrowright F_n(M)$

Definition (Unordered configuration space)

$$\begin{array}{c} F_n(M) \\ \downarrow \\ C_n(M) := F_n(M)/S_n \end{array}$$



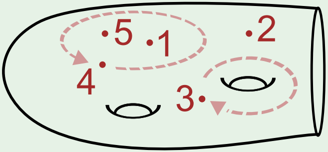
The *unordered configuration space of M on n points* is

$$C_n(M) = \left\{ \begin{array}{c} n\text{-element} \\ \text{subsets of } M \end{array} \right\}$$

Homology of configuration spaces

Goal: Understand $H_*(F_n(M))$.

$S^1 \times S^1 \to F_5(M)$



$\rightsquigarrow$ A class in $H_2(F_5(M))$ (up to sign)

Hard problem: Understand additive relations between these classes.

Key: Fix M.

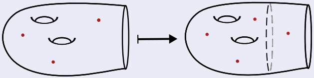
Package $\{H_*(F_n(M))\}_n$ into a single algebraic object, with additional structure coming from S_n–actions and topological operations.

Classical Homological Stablility for $C_n(M)$

M – connected, noncompact manifold of finite type, dimension $\geqslant 2$

$\exists$ *stabilization map*

$t : C_n(M) \to C_{n+1}(M)$



Theorem (McDuff, Segal, 70s))

Fix M.

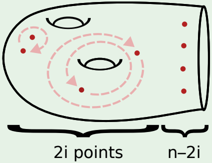
$\{C_n(M)\}_n$ *is* **homologically stable**.

For each i, the maps

$$t_* : H_i(C_n(M)) \to H_i(C_{n+1}(M))$$

is an isomorphism for $n \geqslant 2i$.

$H_i(C_n(M))$ is spanned by:



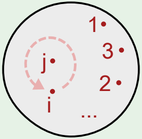
Homological Stablility for $F_n(M)$?

M – connected, noncompact manifold of finite type, dimension $\geqslant 2$

Question: Is $\{F_n(M)\}_n$ homologically stable?

Answer: No!

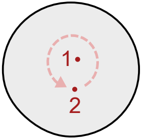
Eg,
$H_1(F_n(D^2)) = \mathbb{Z}^{\binom{n}{2}}$, generators $\alpha_{i,j} =$



$\in H_1(F_n(D^2))$

Up to action of S_n
and stabilization map t,
$\{H_1(F_n(D^2))\}_n$ is generated by:

$\alpha_{1,2} =$



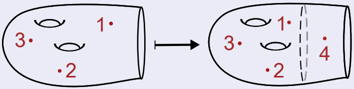
$\in H_1(F_2(D^2))$

Representation Stability for $F_n(M)$

M – connected, noncompact manifold of finite type, dimension $\geqslant 2$

$\exists$ stabilization map

$$t : F_n(M) \to F_{n+1}(M)$$

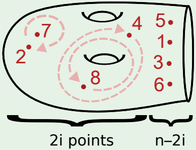


Theorem (Church–Ellenberg–Farb, Miller–W (non-orientable M))

Fix M. For each fixed i, $\{H_i(F_n(M))\}_n$ *is* **representation stable**.

$$\mathbb{Z}[S_{n+1}] \cdot t_*(H_i(F_n(M);\mathbb{Z})) = H_i(F_{n+1}(M);\mathbb{Z}) \qquad \textit{for } n \geqslant 2i.$$

$H_i(F_n(M))$ is spanned by:



Further work

Original results: Church (2012), Church–Ellenberg–Farb (2015)

Generalizations, such as broader classes of spaces *M*, improved stable ranges, alternate stabilization maps, "higher-order" stability:

Church–Ellenberg–Farb–Nagpal (2014)
Ellenberg–Wiltshire-Gordon (2015)
Hersh–Reiner (2017)
Church–Miller–Nagpal–Reinhold (2017)
Moseley–Proudfoot–Young (2017)
Lütgehetmann (preprint)
Tosteson (preprint)
Miller–W (preprint)

Palmer (2013)
Kupers–Miller (2015)
Petersen (2017)
Ramos (2017)
Ramos (2018)
Schiessl (preprint)
Bahran (preprint)
Miller–W (preprint)

Stronger versions & consequences of Theorem

Theorem

Fix M. For each fixed i, $\{H_i(F_n(M))\}_n$ *is* **representation stable***.*

- **finite generation**

$$\mathbb{Z}[S_{n+1}] \cdot t_*(H_i(F_n(M);\mathbb{Z})) = H_i(F_{n+1}(M);\mathbb{Z}) \qquad \text{for } n \geqslant 2i.$$

- **polynomial Betti numbers**

$$dim_{\mathbb{Q}} H_i(F_n(M);\mathbb{Q}) = \text{polynomial in } n \text{ of degree} \leqslant 2i$$

Eg, $\dim_{\mathbb{Q}} H_1(F_n(D^2);\mathbb{Q}) = \binom{n}{2} = \frac{(n)(n-1)}{2}$

Stronger versions & consequences of Theorem

Theorem

Fix M. For each fixed i, $\{H_i(F_n(M))\}_n$ is **representation stable**.

- **multiplicity stability**

  The decomposition of $H_i(F_n(M);\mathbb{Q})$ into irreducible S_n–reps stabilizes for $n \geqslant 4i$.

$$
\begin{aligned}
Eg,\ H_1(F_2(D^2);\mathbb{Q}) &\cong V_{(2)} \\
H_1(F_3(D^2);\mathbb{Q}) &\cong V_{(3)} \oplus V_{(2,1)} \\
H_1(F_4(D^2);\mathbb{Q}) &\cong V_{(4)} \oplus V_{(3,1)} \oplus V_{(2,2)} \\
H_1(F_5(D^2);\mathbb{Q}) &\cong V_{(5)} \oplus V_{(4,1)} \oplus V_{(3,2)} \\
H_1(F_6(D^2);\mathbb{Q}) &\cong V_{(6)} \oplus V_{(5,1)} \oplus V_{(4,2)} \\
H_1(F_7(D^2);\mathbb{Q}) &\cong V_{(7)} \oplus V_{(6,1)} \oplus V_{(5,2)}
\end{aligned}
$$

Stronger versions & consequences of Theorem

Theorem

Fix M. For each fixed i, $\{H_i(F_n(M))\}_n$ is **representation stable**.

- **character polynomials**

  The character $\chi_{H_i(F_n(M);\mathbb{Q})}$ is a polynomial in the "cycle-counting" functions, independent of n.

Eg, $\chi_{H_1(F_n(D^2);\mathbb{Q})}(\sigma) = (\#\text{2-cycles in } \sigma) + \binom{\#\text{1-cycles in } \sigma}{2}$

for $\sigma \in S_n$, for all n.

Stronger versions & consequences of Theorem

Theorem

Fix M. For each fixed i, $\{H_i(F_n(M))\}_n$ *is* **representation stable***.*

- **recursive resolutions**

  For $n \geqslant 2i + 2$*, the* S_n*–rep* $H_i(F_n(M))$ *is determined by a partial resolution by* S_n*–reps*

$$Ind_{S_{n-2}}^{S_n} H_i(F_{n-2}(M)) \longrightarrow Ind_{S_{n-1}}^{S_n} H_i(F_{n-1}(M)) \longrightarrow H_i(F_n(M)) \longrightarrow 0$$

Stronger versions & consequences of Theorem

Theorem

Fix M. For each fixed i, $\{H_i(F_n(M))\}_n$ *is* **representation stable**.

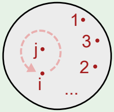
- **free module structure**

  $H_i(F_n(M))$ *is an induced module of a certain form, induced specific from certain subreps of*

$$H_i(F_0(M)), H_i(F_1(M)), \ldots, H_i(F_{2i}(M))$$

Eg, $H_1(F_n(D^2)) = \bigoplus_{\{i,j\}\subseteq\{1,2,\ldots,n\}} \mathbb{Z}\alpha_{i,j}$

$$\cong \mathrm{Ind}_{S_2\times S_{n-2}}^{S_n} H_1(F_2(D^2))$$

$\alpha_{i,j} =$ 

Other instances of representation stability

Analogous behaviour has been established in the (co)homology of:

- certain flag varieties (Weyl group reps)
- hyperplane arrangements associated to reflection groups W_n (W_n-reps)
- $\mathrm{Aut}(F_n)$ and related groups (S_n-reps, etc)
- congruence subgroups $\mathrm{GL}_n(A, I) \subseteq \mathrm{GL}_n(A)$ (S_n- or $\mathrm{GL}_n(A/I)$-reps)
- mapping class groups and moduli spaces (S_n-reps)
- Torelli and related groups ($\mathrm{Sp}_{2g}(\mathbb{Z})$-reps, etc)

⋮

Question: What underlying structure is driving these stability patterns?

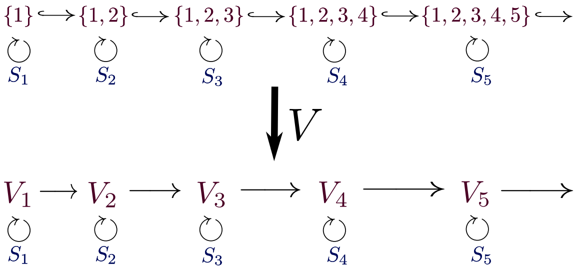
FI and FI-modules

Answer: They are finitely presented FI–modules.

Definition (FI and FI-modules)

Let FI denote the category of **F**inite sets and **I**njective maps.

An FI-*module* is a functor V : FI $\to$ <u>Ab Gps</u>.



Examples of FI-modules

Example: $\mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \cdots$ trivial S_n-reps

Example: $\mathbb{Z} \hookrightarrow \mathbb{Z}^2 \hookrightarrow \mathbb{Z}^3 \hookrightarrow \cdots$ canonical S_n permutation reps

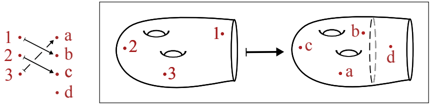
Example: $\mathbb{Z}[x_1] \hookrightarrow \mathbb{Z}[x_1, x_2] \hookrightarrow \mathbb{Z}[x_1, x_2, x_3] \hookrightarrow \cdots$ S_n permutes variables

Non-Example: $\mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \cdots$ alternating S_n-reps

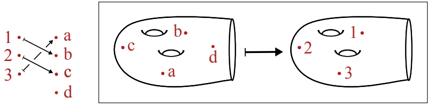
Non-Example: $\mathbb{Z}[S_1] \hookrightarrow \mathbb{Z}[S_2] \hookrightarrow \mathbb{Z}[S_3] \hookrightarrow \cdots$ left regular S_n-reps

Example: $H_i(F_1(M)) \to H_i(F_2(M)) \to H_i(F_3(M)) \to \ldots$

FI-action:



FI^{op}-action:



Finite generation

Finite generation

Homogeneous degree-2 polynomials in $\mathbb{Z}[x_1, x_2, \ldots, x_n]$.

$$
\begin{array}{ccccccc}
S_1 & & S_2 & & S_3 & & \\
\circlearrowright & & \circlearrowright & & \circlearrowright & & \\
R[x_1]_{(2)} & \hookrightarrow & R[x_1, x_2]_{(2)} & \hookrightarrow & R[x_1, x_2, x_3]_{(2)} & \hookrightarrow & \\
\| & & \| & & \| & & \\
\langle x_1^2 \rangle & & \langle x_1^2, x_2^2, x_1x_2 \rangle & & \langle x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_3x_2 \rangle & & \\
\Cup & & \Cup & & & & \\
x_1^2 & & x_1x_2 & & & &
\end{array}
$$

Generators

$\{\mathbb{Z}[x_1, \ldots, x_n]_{(2)}\}_n$ is *finitely generated in degree* $\leqslant 2$ by generators

$$x_1^2 \in V_1, \quad x_1x_2 \in V_2.$$

Current directions

Goals:

- Develop commutative algebraic tools for proving finiteness properties of FI-modules.
- Adapt tools to study other categories (eg) encoding actions of different families of groups.

Thank you!