# Representation Stability and FI–Modules

## Exposition on work by Church – Ellenberg – Farb

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# Background: Classical Homological Stability

$\{Y_n\}_n$ is a sequence of groups or topological spaces,

with inclusions

$$\phi_n \ : \ Y_n \to Y_{n+1}$$

**Definition (Homological Stability)**

The sequence $\{Y_n\}$ is *homologically stable* (over a ring $R$) if for each $k \geq 1$, the map

$$(\phi_n)_* \ : \ H_k(Y_n; R) \to H_k(Y_{n+1}; R)$$

is an isomorphism for $n >> k$.

# Examples of Homologically Stable Sequences

- (Nakaoka 1961)
  **Symmetric groups** $S_n$
- (Arnold 1968, Cohen 1972)
  **Braid groups** $B_n$
- (McDuff 1975, Segal 1979)
  **Configuration spaces of open manifolds**
- (Charney 1979, Maazen 1979, van der Kallen 1980)
  **Linear groups, arithmetic groups (such as** $SL_n(\mathbb{Z})$**)**
- (Harer 1985)
  **Mapping class groups of surfaces with boundary**
- (Hatcher 1995)
  **Automorphisms of free groups** $\mathbf{Aut}(F_n)$
- (Hatcher–Vogtmann 2004)
  **Outer automorphisms of free groups** $\mathbf{Out}(F_n)$

# Generalizing Homological Stability

**What can we say when $H_k(Y_n; R)$ does not stabilize?**

More generally, let $\{V_n\}_n$ be a sequence of $R$-modules.
Suppose $V_n$ has an action by a group $G_n$.

Our objective: A notion of stability for $\{V_n\}_n$ that takes into account the $G_n$–symmetries.

In this talk:

- $G_n = S_n$, the symmetric group
- $R = \mathbb{Q}$, and $V_n$ are finite dimesional vector spaces

# An Example: The Permutation Representation

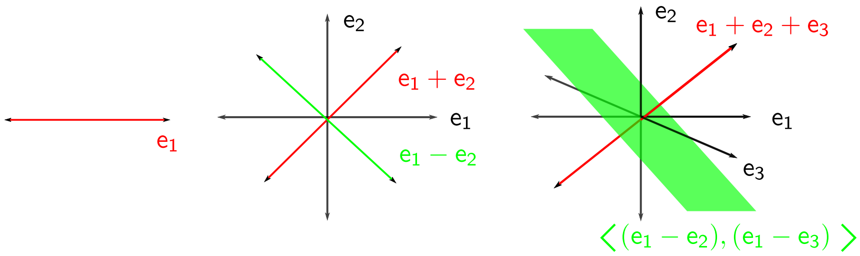
## Example (The Permutation Representation)

Consider the permutation representation

$$V_n = \mathbb{Q}^n = \langle e_1, \ldots, e_n \rangle.$$

For each $n$, $V_n$ decomposes into two irreducibles:

$$\mathbb{Q}^n = \Big\{ a(e_1 + e_2 + \ldots + e_n) \Big\} \oplus \Big\{ a_1 e_1 + \ldots + a_n e_n \;\Big|\; \sum a_i = 0 \Big\}$$



# An Example: The Permutation Representation

**Some properties of the permutation representation**

- The decomposition into irreducibles 'looks the same' for every $n$.

$$\mathbb{Q}^n = \Big\{ a(e_1 + e_2 + \ldots + e_n) \Big\} \oplus \Big\{ a_1 e_1 + \ldots + a_n e_n \ \Big| \ \sum a_i = 0 \Big\}$$

- The dimension of $V_n$ grows polynomially in $n$

$$\dim(V_n) = n$$

- The characters $\chi_n$ of $V_n$ have a 'nice' global description

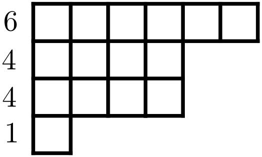
$$\chi_n(\sigma) = \#1\text{–cycles of } \sigma \qquad \text{for all } \sigma \in S_n, \text{ for all } n.$$

**Some facts about $S_n$–representations over $\mathbb{Q}$**

- Every $S_n$–representation decomposes uniquely as a sum of irreducibles.

$$\lambda = (6, 4, 4, 1)$$

- Irreducibles are indexed by partitions $\lambda$ of $n$, depicted by *Young diagrams*.



## Obstacle

How can we compare irreducibles for different values of $n$?

## Solution

Two irreducibles are “the same” if only the top rows of their Young diagrams differ.

## Example (The Permutation Representation $V_n = \mathbb{Q}^n$)

$$\mathbb{Q}^1 = V_{\ydiagram{1}}$$

$$\mathbb{Q}^2 = V_{\ydiagram{2}} \oplus V_{\ydiagram{1,1}}$$

$$\mathbb{Q}^3 = V_{\ydiagram{3}} \oplus V_{\ydiagram{2,1}}$$

$$\mathbb{Q}^4 = V_{\ydiagram{4}} \oplus V_{\ydiagram{3,1}}$$

$$\mathbb{Q}^5 = V_{\ydiagram{5}} \oplus V_{\ydiagram{4,1}}$$

# The Definition of an FI–module

**Definition (Church–Ellenberg–Farb) (The Category FI)**

Denote by FI the category of Finite sets with Injective maps

$$\{1\} \hookrightarrow \{1,2\} \hookrightarrow \{1,2,3\} \hookrightarrow \{1,2,3,4\} \hookrightarrow \{1,2,3,4,5\} \hookrightarrow$$

$$\circlearrowright \qquad \circlearrowright \qquad \circlearrowright \qquad \circlearrowright \qquad \circlearrowright$$

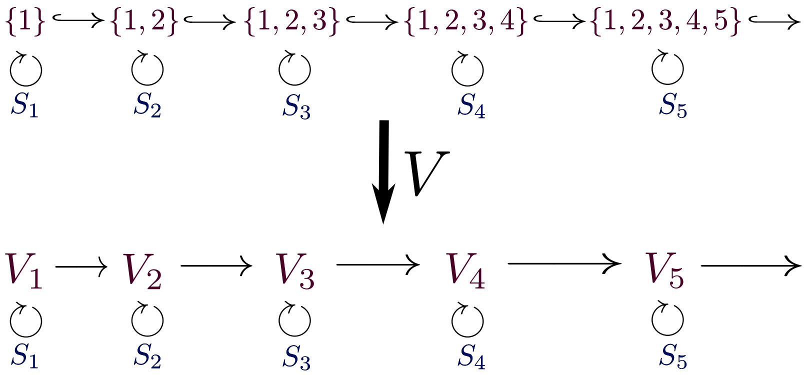
$$S_1 \qquad S_2 \qquad S_3 \qquad S_4 \qquad S_5$$

# The Definition of an FI–module

**Definition (Church–Ellenberg–Farb) (FI–Modules)**

A (rational) *FI–module* is a functor

$$V : \mathrm{FI} \to \mathbb{Q}\text{-Vect}$$



# Finite Generation of FI–Modules

## Definition (Generation)

If $V$ is an FI–module, and $S \subseteq \coprod_n V_n$, then the *FI–module generated by $S$* is the smallest sub–FI–module containing the elements of $S$.

## Definition (Finite Generation)

An FI–module is *finitely generated* if it has a finite generating set.

## Example (The Permutation Representation $V_n = \mathbb{Q}^n$)

The permutation representation $V_n = \mathbb{Q}^n = \langle e_1, \ldots, e_n \rangle$ is generated by $e_1 \in V_1$.

# Consequences of Finite Generation

**Theorem (Church–Ellenberg–Farb)**

*Let $V$ be a finitely-generated FI–module. Then for $n >> 1$*

- *The decomposition into irreducible $S_n$–representations stabilizes.*
- *$\dim(V_n)$ is polynomial in $n$*
- *The characters $\chi_n$ of $V_n$ are given by a (unique) polynomial in the variables $X_r$*

$$X_r(\sigma) = \#r\text{–cycles of } \sigma \qquad \text{for all } \sigma \in S_n, \text{ for all } n.$$

*Any sub–FI–module of $V$ also has these properties.*

We call the sequence $\{V_n\}_n$ *uniformly representation stable*.

# Some Representation Stable Cohomology Sequences

| | | |
|---|---|---|
| (Church–Farb) | $\{H^k(P_n;\mathbb{Q})\}_n$ | **The pure braid group** |
| (Jimenez-Rolland) | $\{H^k(\mathrm{PMod}(\Sigma^n_{g,r});\mathbb{Q})\}_n$ | **The pure MCG of an *n*-puncture surface** $\Sigma^n_{g,r}$ |
| (Church) | $\{H^k(\mathrm{PConf}_n(M);\mathbb{Q})\}_n$ | **Ordered configuration space of a manifold** $M$ |
| (Putman) | $\{H^k(\mathrm{PMod}^n(M);\mathbb{F})\}_n$ | **The pure MCG of an *n*-puncture manifold** |
| (Putman) | Eg, $\{H^k(\mathrm{SL}_n(\mathbb{Z},\ell);\mathbb{F})\}_n$ | **Certain congruence subgroups** |
| (Wilson) | $\{H^k(P\Sigma_n;\mathbb{Q})\}_n$ | **The pure symmetric automorphism group** $P\Sigma_n$ **of the free group** |

# Open Question

**Problem**

Compute the characters, and the stable decompositions into irreducibles, in the above examples.

# My Research

**My current project**

To develop a unified “FI–module theory” for the three families of classical Weyl groups.

# References

## Further Reading


T Church, B Farb.
*Representation theory and homological stability*, preprint, 2010.

T Church, J Ellenberg, B Farb.
*FI–modules: A new approach to stability for $S_n$–representations*, preprint, 2012.

The End

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