COGENT Summer School 2022

The high-degree rational cohomology of $\mathbf{SL}_n(\mathbb{Z})$ and its principal congruent subgroups

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Abstracts

In Wilson's 3-part lecture series we will survey some classical and recent results on the high-degree rational cohomology of $\mathrm{SL}_n(\mathbb{Z})$, or more generally $\mathrm{SL}_n(R)$ when R is a number ring. These cohomology groups are governed by an $\mathrm{SL}_n(R)$-representation called the Steinberg module. We will discuss how we can study the groups $H^*(\mathrm{SL}_n(R);\mathbb{Q})$ by studying the topology of certain simplicial complexes associated to the Steinberg modules.

Contents

1 Patzt Lecture 1: Borel–Serre Duality

In this first lecture, we will introduce the main players of this lecture series, $\mathrm{SL}_n\mathbb{Z}$ and its congruence subgroups. We proceed to look at their symmetric spaces that will help us compute the group cohomology of the main players. We furthermore discuss the Borel–Serre bordification that is necessary to prove the central method to analyze the high dimensional cohomology of theses groups: Borel–Serre duality.

The main players: $\mathrm{SL}_n\mathbb{Z}$ shall denote the integer $n \times n$ matrices with determinant 1. Its prime p level principal congruence subgroups are

$$\Gamma_n(p) := \{A \in \mathrm{SL}_n\mathbb{Z} \mid A \equiv I_n \mod p\}.$$

These groups are simultaneously the kernels of the $\mod p$ surjection

$$\mathrm{SL}_n\mathbb{Z} \longrightarrow \mathrm{SL}_n\mathbb{F}_p.$$

Goal: In this lecture series, we will investigate $H^i(\mathrm{SL}_n\mathbb{Z};\mathbb{Q})$ and $H^i(\Gamma_n(p);\mathbb{Z})$ for i large. We will see later in this lecture that these cohomology groups are zero if $i > \binom{n}{2}$. Therefore "large" means that i is close below $\binom{n}{2}$.

Symmetric spaces: The main tool that allows us to study the group cohomology of $\mathrm{SL}_n\mathbb{Z}$ and its congruence subgroups is its symmetric space. In particular, that is the quotient space

$$X_n = \mathrm{SL}_n\mathbb{R}/\mathrm{SO}(n).$$

Here, we use the topology of $\mathrm{SL}_n\mathbb{R}$ as a subset of $\mathbb{R}^{n^2}$ and the quotient by collapsing the cosets modulo $\mathrm{SO}(n)$ to points. This space is homeomorphic to the space of positive definite symmetric real $n \times n$ matrices by sending the coset of a matrix $A \in \mathrm{SL}_n\mathbb{R}$ to the symmetric matrix AA^T. This space in fact is an $\left(\frac{n(n+1)}{2} - 1\right)$-dimensional manifold.

Consider that action of $\mathrm{SL}_n\mathbb{Z}$ on X_n by multiplication on the left. This action is properly discontinuous but not free. As a matter of fact it has finite stabilizers. For example consider, the stabilizer of the coset of the identity matrix I_n. It is $\mathrm{SL}_n\mathbb{Z} \cap \mathrm{SO}(n)$. But there are only finitely many unit vectors in $\mathbb{Z}^n$ that the standard basis could be sent to.

This observation shows that for all subgroups $\Gamma \leq \mathrm{SL}_n\mathbb{Z}$, we can compute rational group cohomology with the symmetric spaces, as

$$H^*(\Gamma;\mathbb{Q}) \cong H^*(X_n/\Gamma;\mathbb{Q}).$$

For torsion free subgroups Γ (for example $\Gamma_n(p)$ with $p > 2$), we don't have any stabilizers anymore, so

$$H^*(\Gamma;\mathbb{Z}) \cong H^*(X_n/\Gamma;\mathbb{Z}).$$

Borel–Serre bordification: Both X_n and X_n/Γ are non-compact manifolds. Borel and Serre constructed a bordification $\overline{X}_n$ such that $\overline{X}_n/\Gamma$ is a compact manifold with boundary (and corners if considered differentially) if Γ is a finite index subgroup of $\mathrm{SL}_n\mathbb{Z}$. This helps us because we can apply Poincaré duality to $\overline{X}_n/\Gamma$.

There are multiple ways to add boundary to X_n. The way that Borel and Serre did it has multiple advantage for us. One is that the $\mathrm{SL}_n\mathbb{Z}$ action of X_n extends to an action on $\overline{X}_n$. The second is that $X_n = \overline{X}_n \setminus \partial\overline{X}_n$ embeds into $\overline{X}_n$ as a homotopy equivalence, and so does X_n/Γ into $\overline{X}_n/\Gamma$ if Γ is a finite index subgroup of $\mathrm{SL}_n\mathbb{Z}$. In particular, we can compute the group cohomology of Γ by

$$H^*(\Gamma;\mathbb{Q}) \cong H^*(X_n/\Gamma;\mathbb{Q}) \cong H^*(\overline{X}_n/\Gamma;\mathbb{Q}).$$

(And even integrally if Γ is torsion free.)

A neat way to construct $\overline{X}_n$ is actually as a subspace of X_n by removing slightly shrunk horoballs. Let us look at this in the example of $n = 2$:

$X_2 \cong \mathbb{H}_2$ the complex upper half plane (i.e. all $z \in \mathbb{C}$ with $\operatorname{im} z > 0$). Such a homeomorphism is given by sending the coset of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\mathbb{R}$ to

$$z = \frac{ai+b}{ci+d} = \frac{(ai+b)(-ci+d)}{c^2+d^2} = \frac{(ad-bc)i+(ac+bd)}{c^2+d^2} = \frac{i+(ac+bd)}{c^2+d^2}.$$

Note that $A \in \mathrm{SO}(2)$ if and only if $ac+bd = 0$ and $c^2+d^2 = 1$ or equivalently $z = i$.

In Figure 1, horoballs are indicated. These are circles that touch the real line in the rational numbers $q \in \mathbb{Q}$ and if $q = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ fully reduced, they have diameter b^2. Additionally there is one horizontal line that goes through i. It should be though of as the horoball touching infinity. One can observe that none of these circles intersect but a lot of them touch. It is also true that $\mathrm{SL}_2\mathbb{Z}$ sends a horoball to another horoball.

If we now slightly decrease the radius of each circle but keep the attachment points fixed and then remove their interior from $\mathbb{H}_2$, we get a connected manifold with boundary. Note that this manifold is still contractible and non-compact.

Its boundary is a disjoint union of open intervals, one of each horoball, so one for every rational number and one for infinity. This is homotopy equivalent to the discrete set $\mathbb{Q} \cup \{\infty\}$ which we can view as the set of lines in $\mathbb{Q}^2$ by sending

$$\begin{aligned} q \in \mathbb{Q} &\longmapsto \begin{pmatrix} q \\ 1 \end{pmatrix} \\ \infty &\longmapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

In general, the boundary $\partial\overline{X}_n$ is homotopy equivalent to the Tits building $\mathcal{T}_n(\mathbb{Q})$ that we introduce next. Note that $\partial\overline{X}_n$ is not homeomorphic to the Tits building. One further

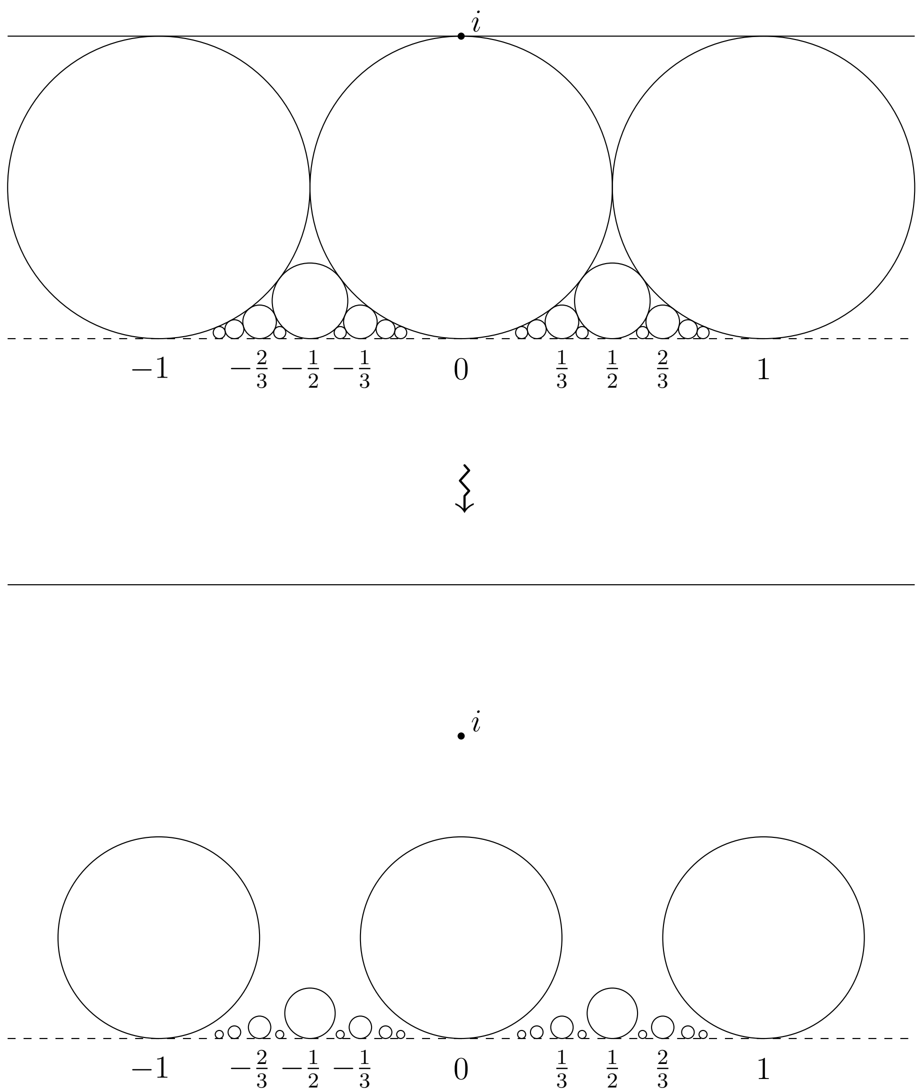


Figure 1: Shinking Horoballs

observes that $\partial\overline{X}_n/\Gamma$ is not homotopy equivalent to $\mathcal{T}_n(\mathbb{Q})/\Gamma$. But this will not directly play a role in this lecture series.

Tits building: Let F be a field. $\mathcal{T}_n(F)$ is the simplicial complex that has nonzero proper subspaces of F^n as its vertices and every flag

$$0 \subsetneq V_0 \subsetneq \cdots \subsetneq V_p \subsetneq F^n$$

forms a p-simplex.

Solomon–Tits: $T_n(F)$ is a wedge of $(n-2)$-spheres. In particular, it only has reduced homology in dimension $n-2$.

Steinberg module: The Steinberg module is the free abelian group whose basis are the $(n-2)$-spheres: $\mathrm{St}_n F := \widetilde{H}_{n-2}(\mathcal{T}_n(F);\mathbb{Z})$. It comes with an action of $\mathrm{SL}_n F$ on it.

Borel–Serre duality: This is the crucial tool that lets us access high dimensional cohomology of $\mathrm{SL}_n\mathbb{Z}$ and its finite index subgroups. Let Γ be a finite index subgroup of $\mathrm{SL}_n\mathbb{Z}$. Then the following isomorphism is true.

$$H^{\binom{n}{2}-i}(\Gamma;\mathbb{Q}\otimes M) \cong H_i(\Gamma;\mathbb{Q}\otimes M\otimes \mathrm{St}_n\mathbb{Q})$$

This isomorphism holds even integrally if Γ is torsion free:

$$H^{\binom{n}{2}-i}(\Gamma;M) \cong H_i(\Gamma;M\otimes \mathrm{St}_n\mathbb{Q})$$

Corollary:

$$H^k(\Gamma;\mathbb{Q}\otimes M) \cong 0 \quad \text{for } k > \binom{n}{2}.$$

This statement can be generalized to other number rings. More about that in the next lecture by Jenny.

General duality groups: A group G is called a *(virtual) duality group* of dimension ν with *dualizing module* D if

$$H^{\nu-i}(G;(\mathbb{Q}\otimes)M) \cong H_i(G;(\mathbb{Q}\otimes)M\otimes D)$$

for all G-modules and $i\in\mathbb{N}$.

Observe that this implies that

$$H^i(G;(\mathbb{Q}\otimes)M) \cong 0$$

for all G-modules and $i>\nu$.

Bieri–Eckmann: G is a duality group of dimension ν if and only if

$$H^k(G;\mathbb{Z}G) \cong 0 \quad \text{for } k \neq \nu$$

and

$$H^\nu(G;\mathbb{Z}G) \text{ is free abelian.}$$

G is a virtual duality group of dimension ν if and only if

$$H^k(G;\mathbb{Q}G) \cong 0 \quad \text{for } k \neq \nu.$$

Note that if G is a dualizing group of dimension ν, the dualizing module is uniquely determined:

$$H^\nu(G;\mathbb{Z}G) \cong H_0(G;\mathbb{Z}G \otimes D) = (\mathbb{Z}G \otimes D)_G \cong D$$

Proof of Borel–Serre duality: Let Γ be a finite index subgroup of $\mathrm{SL}_n\mathbb{Z}$. For simplicity, we assume that Γ is torsion free. (Otherwise rationalize all coefficients.)

We will apply the theorem of Bieri and Eckmann. We can compute

$$H^k(\Gamma;\mathbb{Z}\Gamma) \cong H^k(\overline{X}_n/\Gamma;\mathbb{Z}\Gamma)$$

using Poincaré duality and see that it is isomorphic to the relative homology

$$H_{\frac{n(n+1)}{2}-1-k}(\overline{X}_n/\Gamma, \partial\overline{X}_n/\Gamma;\mathbb{Z}\Gamma).$$

Using a version of Shapiro's lemma, this isomorphic to

$$H_{\frac{n(n+1)}{2}-1-k}(\overline{X}_n, \partial\overline{X}_n;\mathbb{Z})$$

which is isomorphic to

$$\widetilde{H}_{\frac{n(n+1)}{2}-1-k-1}(\partial\overline{X}_n;\mathbb{Z})$$

using the long exact sequence of a pair together with the fact that $\widetilde{X}_n$ is contractible.

As we have noted above, $\partial\overline{X}_n$ is homotopy equivalent to the Tits building $\mathcal{T}_n(\mathbb{Q})$ and thus only has reduced homology if

$$\frac{n(n+1)}{2} - 1 - k - 1 = n - 2$$

which is exactly when $k = \binom{n}{2}$.

It remains to prove that

$$H^k(\Gamma;\mathbb{Z}\Gamma) \cong \widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q});\mathbb{Z}) = \mathrm{St}_n\mathbb{Q}$$

is free abelian. This follows because $\mathcal{T}_n(\mathbb{Q})$ is $(n-2)$-dimensional and thus $\mathrm{St}_n\mathbb{Q}$ is the kernel of the map on simplicial chains

$$\widetilde{C}_{n-2}(\mathcal{T}_n(\mathbb{Q})) \longrightarrow \widetilde{C}_{n-3}(\mathcal{T}_n(\mathbb{Q})).$$

Church–Farb–Putman conjecture: Finally, we will state a conjecture about the high dimensional cohomology of $\mathrm{SL}_n\mathbb{Z}$ that the next four lectures will focus on:

$$H^{\binom{n}{2}-i}(\mathrm{SL}_n\mathbb{Z};\mathbb{Q}) \cong 0 \quad \text{for } i \leq n-2.$$

2 Wilson Lecture 1: The Steinberg module, (integral) apartment classes, and the top-degree cohomology of SL$_n(R)$

Throughout this lecture, we will use the following notation.

- Let n be an integer, which we almost always assume to be at least 2.
- Let F denote a number field, that is, a finite field extension of $\mathbb{Q}$.
- Let R denote the ring of integers in F, that is, the solutions in F to all monic polynomials with coefficients in $\mathbb{Z}$.

Exercise 1. Verify that F is the field of fractions of R.

Our goal is to say something about the cohomology of $\mathrm{SL}_n(R)$. We are primarily interested in the case that the number ring $R \subseteq F$ is the integers $\mathbb{Z} \subseteq \mathbb{Q}$, but you may also keep in mind examples like the Gaussian integers $\mathbb{Z}[i] \subseteq \mathbb{Q}(i)$, the Eisenstein integers $\mathbb{Z}[\frac{1}{2}(-1+i\sqrt{3})] = \mathbb{Z}[e^{2\pi i/6}] \subseteq \mathbb{Q}(\sqrt{3})$, the cyclotomic integers $\mathbb{Z}[e^{2\pi i/d}] \subseteq \mathbb{Q}(e^{2\pi i/d})$, etc.

Virtual cohomological dimension

In this lecture series, we will assume the basic results on the cohomology of groups, including the concepts of *(virtual) cohomological dimension* and *Bieri–Eckmann duality*. Although we aim to blackbox the results we use, students are encouraged to look at the accompanying review package on homological algebra and cohomology of groups for this background.

The results that Peter outlined in the first lecture all generalize from $\mathrm{SL}_n(\mathbb{Z})$ to $\mathrm{SL}_n(R)$. Peter wrote ν to denote the *virtual cohomological dimensions* (vcd) of these groups, a value computed for $\mathrm{SL}_n(R)$ by Borel and Serre.

Theorem 2.1 (Borel–Serre). *Let R be a number ring with fraction field F. The virtual cohomological dimension (vcd) of $SL_n(R)$ is*

$$vcd_n(R) = r\binom{n+1}{2} + cn^2 - n - r - c + 1$$

where

- *r is the number of embeddings $F \hookrightarrow \mathbb{R}$*
- *c is the number of pairs of complex embeddings $F \hookrightarrow \mathbb{C}$ that do not factor through $\mathbb{R}$*

This result implies, notably, that the vcd of $\mathrm{SL}_n(R)$ is known; it is quadratic in n, and it depends on the Galois theory of F. In particular, when $R = \mathbb{Z}$, then the vcd is $\binom{n}{2}$.

Readers can refer to the review package on cohomology of groups for the definition of virtual cohomological dimension of a group with torsion-free finite index subgroups. For our purposes, crucially, this number has the property that it bounds the cohomological dimension of $\mathrm{SL}_n(R)$ with (possibly twisted) rational coefficients. In fact,

$$vcd_n(R) = \max\{\ q \mid H^q(\mathrm{SL}_n(R); V) \neq 0 \text{ for some } \mathbb{Q}[SL_n(R)]\text{-module } V\}.$$

Consider the cohomology of $\mathrm{SL}_n(R)$ with (trivial, untwisted) coefficients in $\mathbb{Q}$. Although there is *some* rational $\mathrm{SL}_n(R)$-representation V such that $H^{vcd_n}(\mathrm{SL}_n(R); V) \neq 0$, the definition of vcd does not address the question of whether $H^{vcd_n}(\mathrm{SL}_n(R); \mathbb{Q})$ vanishes, and in fact this question is open for many rings R.

The following problem is open for many number rings R, including even for $R = \mathbb{Z}$, and it is one of the motivating questions of this talk series.

Open Problem 2.2. *Let R be a number ring. For each n, what is the largest value of q such that*

$$H^q(SL_n(R); \mathbb{Q}) \neq 0?$$

More generally, how does the answer to this problem depend on ring-theoretic properties of R?

Virtual Bieri–Eckmann duality

Recall from Peter's lecture that (rationally) the groups $\mathrm{SL}_n(R)$ satisfy a twisted analogue of Poincaré duality, called *virtual Bieri–Eckmann duality*.

Theorem 2.3 (Borel–Serre). *Let R be a number ring with fraction field F. Let V be a rational $SL_n(R)$-representation. There are, for each n, isomorphisms*

$$H^{vcd_n - i}(SL_n(R); V) \cong H_i(SL_n(R); V \otimes_{\mathbb{Z}} St_n(F))$$

where $St_n(F)$ is the Steinberg module associated to $SL_n(R)$.

We informally call $H^{vcd_n - i}(\mathrm{SL}_n(R); \mathbb{Q})$ the *codimension-i* cohomology of $\mathrm{SL}_n(R)$. This result shows that, to study the high-degree rational cohomology of $\mathrm{SL}_n(R)$, we can instead study the low-degree homology, at the expense of working with twisted coefficients.

Recall that this theorem follows from work of Borel and Serre on the *bordification* of the symmetric space associated to $\mathrm{SL}_n(R)$. The Steinberg module $\mathrm{St}_n(F)$ is defined in terms of the *Tits building* $\mathcal{T}_n(F)$, a simplicial complex we can identify (up to homotopy equivalence) with the boundary of this bordification.

Definition 2.4. Fix a field F. The Tits building $\mathcal{T}_n(F)$ is an abstract simplicial complex defined as follows.

- Vertices V of $\mathcal{T}_n(F)$ correspond to proper nonzero vector subspaces of F^n.
- A collection of vertices $\{V_0, \ldots, V_p\}$ span a p-simplex if and only if (possibly after re-indexing) the subspaces form a flag
$$0 \subsetneq V_0 \subsetneq \cdots \subsetneq V_p \subsetneq F^n.$$

In other words, $\mathcal{T}_n(F)$ is the simplicial complex of chains in the poset of proper nonzero subspaces of F^n under inclusion.

Up to homotopy equivalence, the Tits building is a wedge of spheres. A guided (modern) proof of the following theorem is given in the exercises.

Theorem 2.5 (Solomon–Tits)**.** *Let F be a field. There is a homotopy equivalence $\mathcal{T}_n(F) \simeq \bigvee S^{n-2}$.*

When F is a number field, this wedge consists of a countably infinite number of $(n-2)$-spheres.

Since $\mathrm{SL}_n(R)$ acts on F^n and its set of subspaces, and the action respects inclusion, there is an induced simplicial action of $\mathrm{SL}_n(R)$ on the Tits building. The Steinberg module $\mathrm{St}_n(F)$ is defined to be the single nonvanishing reduced homology group of the Tits building $\mathcal{T}_n(F)$, viewed as a $\mathrm{SL}_n(R)$-representation.

Definition 2.6. Let F be a number field and R its ring of integers. The *Steinberg module* of $\mathrm{SL}_n(R)$ is the $\mathbb{Z}[\mathrm{SL}_n(R)]$-module
$$\mathrm{St}_n(F) := \widetilde{H}_{n-2}(\mathcal{T}_n; \mathbb{Z}).$$

Conjectures and known results high-degree rational cohomology of $\mathbf{SL}_n(R)$

Let R be a number ring and F its field of fractions. Consider the cohomology of $\mathrm{SL}_n(R)$ with trivial rational coefficients $\mathbb{Q}$.

Goal 2.7. Study the cohomology groups $H^q(\mathrm{SL}_n(R);\mathbb{Q})$ when q is close to the vcd.

In the case $R = \mathbb{Z}$, Church–Farb–Putman [CFP1] conjectured that these high-degree cohomology groups in fact vanish in a range close to the vcd.

Conjecture 2.8 (Church–Farb–Putman). *$H^{vcd_n - i}(SL_n(\mathbb{Z});\mathbb{Q}) = 0$ for all $n \geq i+2$.*

An aside: we may frame their conjecture as a form of stability that is in a sense dual to the classical homological stability results for $\mathrm{SL}_n(\mathbb{Z})$ due to Borel. They predict moreover that the stable groups are zero.

It is natural to wonder whether these conjectures also hold for $\mathrm{SL}_n(R)$ for other number rings R. Some known results are summarized in the table below. These results give some support (in low degree) for the Church–Farb–Putman conjecture, and illustrate how the behaviour of these cohomology groups varies depending on the ring-theoretical properties of R. The analogue of the Church–Farb–Putman conjecture may hold for other Euclidean number rings, but these results show that the conjectures cannot hold (without modification) in general.

		Top degree $q = vcd_n$
Lee–Szczarba [LS]	R a Euclidean domain e.g. $R = \mathbb{Z}, \mathbb{Z}[i]$	$H^{vcd_n}(\mathrm{SL}_n(R);\mathbb{Q}) = 0$ for all $n \geq 2$
Church–Farb–Putman [CFP2]	R not a PID e.g. $R = \mathbb{Z}[\sqrt{-5}]$	$H^{vcd_n}(\mathrm{SL}_n(R);\mathbb{Q}) \neq 0$ for all $n \geq 2$
Miller–Patzt–Wilson–Yasaki [MPWY]	$F = \mathbb{Q}(\sqrt{d})$ for $d = -43, -67, -163$	$H^{vcd_{2n}}(\mathrm{SL}_{2n}(R);\mathbb{Q}) \neq 0$ for all $2n \geq 2$

Weinberger proved that, assuming the Generalized Riemann Hypothesis (GRH), the only number rings that are PID but not Euclidean are the rings of integers in the quadratic number fields $F = \mathbb{Q}(\sqrt{d})$ for $d = -19, -43, -67, -163$. Thus, assuming the GRH, the only number ring not addressed in this table is the ring of integers in $\mathbb{Q}(\sqrt{-19})$. In this case it is known that $H^{vcd_2}(\mathrm{SL}_2(R);\mathbb{Q}) = 0$ (see Vogtmann [V, Table 7.1]), and the general question is open. In all examples of non-Euclidean PID's, the case of n odd is open in general.

Codimension 1 $q = vcd_n - 1$		
Church–Putman [CP]	$R = \mathbb{Z}$	$H^{vcd_n-1}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q}) = 0$ for all $n \geq 3$
Kupers–Miller–Patzt–Wilson [KMPW]	R is the Gaussian or Eisenstein integers	$H^{vcd_n-1}(\mathrm{SL}_n(R);\mathbb{Q}) = 0$ for all $n \geq 3$

Earlier this year the codimension-2 case of the Church–Farb–Putman conjectures was announced.

Codimension 2 $q = vcd_n - 2$		
Brück–Miller–Patzt–Sroka–Wilson [BMPSW]	$R = \mathbb{Z}$	$H^{vcd_n-2}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q}) = 0$ for all $n \geq 4$ (in fact for all $n \geq 3$)

Higher-codimension cases of the Church–Farb–Putman Conjecture (Conjecture 2.8), and the broader situation for other PIDs, are open problems. Kupers–Miller–Patzt–Wilson [KMPW] showed that the approach to the proof that they and Church–Putman [CP] took for the codimension 1 case provably will not work for all Euclidean domains. But, this negative result does not disprove the vanishing of $H^{vcd_n-1}(\mathrm{SL}_n(R);\mathbb{Q})$ in these cases, so new ideas are needed.

A general approach to the Church–Farb–Putman conjectures

Consider cohomology with trivial coefficients $\mathbb{Q}$. By virtual Bieri–Eckmann duality (Theorem 2.3), there are isomorphisms

$$H^{vcd_n - i}(\mathrm{SL}_n(R);\mathbb{Q}) \cong H_i(\mathrm{SL}_n(R);\mathrm{St}_n(F)\otimes\mathbb{Q}).$$

This means, using general properties of group (co)homology, one strategy to compute these groups is the following. (See the review package on group cohomology for details).

Recall here that, given a group G and G-representation V, the *coinvariants* V_G are the quotient group

$$V/\langle\, v - g\cdot v \mid v\in V, g\in G\,\rangle,$$

that is, V_G is the largest G-equivariant quotient of V with trivial G-action.

To compute $H^{vcd_n - i}(\mathrm{SL}_n(R);\mathbb{Q}) \cong H_i(\mathrm{SL}_n(R);\mathrm{St}_n(F)\otimes\mathbb{Q})$:

- Find a resolution of $\mathbb{Q}\otimes_{\mathbb{Z}}\mathrm{St}_n(F)$ by flat $\mathbb{Q}[\mathrm{SL}_n(R)]$-modules

$$\ldots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow \mathbb{Q}\otimes_{\mathbb{Z}}\mathrm{St}_n(F) \longrightarrow 0$$

- Take $\mathrm{SL}_n(R)$-coinvariants

$$\ldots \longrightarrow (V_1)_{\mathrm{SL}_n(R)} \longrightarrow (V_0)_{\mathrm{SL}_n(R)} \longrightarrow 0$$

- Take homology of this resulting chain complex. Its ith homology group is isomorphic to $H^{vcd_n - i}(\mathrm{SL}_n(R);\mathbb{Q})$.

Thus, an approach to our Goal 2.7 is to attempt to compute a flat resolution of $\mathbb{Q}\otimes_{\mathbb{Z}}\mathrm{St}_n(F)$ by $\mathrm{SL}_n(R)$-representations that are 'nice' enough that we have a hope of understanding their coinvariants.

When R is a Euclidean domain, a construction of one flat resolution, the *Sharbly resolution* due to Lee–Szczarba [LS], is outlined in the exercises. They used this resolution to prove $H^{vcd_n}(\mathrm{SL}_n(R);\mathbb{Q})$ vanishes for R Euclidean. Unfortunately it appears that this resolution is "too big" to compute $H^{vcd_n - i}(\mathrm{SL}_n(R);\mathbb{Q})$ for $i>0$.

Today's goal: vanishing of $H^{vcd_n}(\mathrm{SL}_n(R);\mathbb{Q})$ for R Euclidean

Assume that our number ring R is a Euclidean domain. For example, R could be $\mathbb{Z}, \mathbb{Z}[i], \mathbb{Z}[\sqrt{2}], \ldots$

The goal of today's talk is to give a proof of Lee–Szczarba's result that $H^{vcd_n}(\mathrm{SL}_n(R);\mathbb{Q})$ vanishes for all $n \geq 2$. Our proof is anachronistic—we will use a result of Ash–Rudolph [AR] (to be proved in the next lectures) that is more recent than Lee–Szczarba [LS].

By Bieri–Eckmann duality (Theorem 2.3),

$$\begin{aligned} H^{vcd_n}(\mathrm{SL}_n(R);\mathbb{Q}) &\cong H_0(\mathrm{SL}_n(R);\mathrm{St}_n(F)\otimes\mathbb{Q}) \\ &\cong (\mathrm{St}_n(F)\otimes\mathbb{Q})_{\mathrm{SL}_n(R)} \end{aligned}$$

Our strategy: to find generators of $\mathrm{St}_n(F)\otimes\mathbb{Q}$ that we can show vanish in $\mathrm{SL}_n(R)$-coinvariants.

Warm-up: The case $n = 2$

When $n = 2$, the Tits building is a discrete set

$$\mathcal{T}_2(F) = \{\text{lines in } F^2\}.$$

Thus the Steinberg module is the representation

$$\mathrm{St}_2(F) = \widetilde{H}_0(\mathcal{T}_2(F)) \cong \langle\, L_1 - L_2 \mid L_i \subseteq F^2 \text{ a line}\,\rangle.$$

Let's specialize to the case $R = \mathbb{Z}$. What can we say about the $\mathrm{SL}_2(\mathbb{Z})$-coinvariants of $\mathrm{St}_2(\mathbb{Q})$? Remember: our goal is to show these coinvariants vanish, which implies

$$H^{vcd_2}(\mathrm{SL}_2(\mathbb{Z});\mathbb{Q}) = 0.$$

Consider the generator $x = \mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} - \mathbb{Q}\begin{bmatrix}0\\1\end{bmatrix}$. Now, consider the matrix

$$g = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

The matrix g interchanges the two coordinate axes, so $g\cdot x = -x$. Conclusion: x vanishes in coinvariants.

That was promising! Is it possible that we can use the same trick for each of our generators $L_1 - L_2$ of $\mathrm{St}_2(F)$?

Unfortunately not. Next consider the generator $y = \mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} - \mathbb{Q}\begin{bmatrix}2\\3\end{bmatrix}$.

Exercise 2. (a) Verify that there is no element $g \in \mathrm{SL}_2(\mathbb{Z})$ that interchanges the lines $\mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbb{Q}\begin{bmatrix}2\\3\end{bmatrix}$.

(b) Explain why the reason we can't adapt the trick we used for $x = \mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} - \mathbb{Q}\begin{bmatrix}0\\1\end{bmatrix}$ is that

$$\left(\mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} \cap \mathbb{Z}^2\right) \oplus \left(\mathbb{Q}\begin{bmatrix}2\\3\end{bmatrix} \cap \mathbb{Z}^2\right) \neq \mathbb{Z}^2.$$

In this case, the $\mathbb{Z}$-module generators $\begin{bmatrix}1\\0\end{bmatrix}$ for $\left(\mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} \cap \mathbb{Z}^2\right)$ and $\begin{bmatrix}2\\3\end{bmatrix}$ for $\left(\mathbb{Q}\begin{bmatrix}2\\3\end{bmatrix} \cap \mathbb{Z}^2\right)$ do not form a basis for $\mathbb{Z}^2$ but instead span an index-3 subgroup.

The crux of our proof that the coinvariants $\mathrm{St}_2(\mathbb{Q})_{\mathrm{SL}_2(\mathbb{Z})}$ vanish is the claim that $\mathrm{St}_2(\mathbb{Q})$ is in fact generated by the subset of generators $L_1 - L_2$ with good integrality properties, the generators for which it is possible to find a matrix g as above. In general, when R is Euclidean,

$$\mathrm{St}_2(R) = \langle\ L_1 - L_2 \mid L_i \subseteq F^2 \text{ a line}, (L_1 \cap R) \oplus (L_2 \cap R) = R^2\ \rangle.$$

In other words, for each generator $L_1 - L_2$ we can write $L_1 = Fv_1$ and $L_2 = Fv_2$ for some basis v_1, v_2 of R^2.

Exercise 3. Write

$$\mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} - \mathbb{Q}\begin{bmatrix}2\\3\end{bmatrix} = \left(\mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} - \mathbb{Q}\begin{bmatrix}1\\1\end{bmatrix}\right) + \left(\mathbb{Q}\begin{bmatrix}1\\1\end{bmatrix} - \mathbb{Q}\begin{bmatrix}2\\3\end{bmatrix}\right).$$

Show that this class vanishes in coinvariants.

We will now see that this strategy generalizes to all $n \geq 2$.

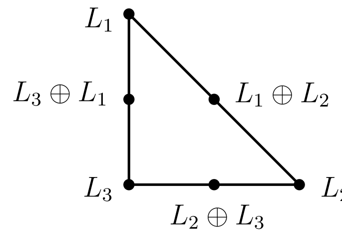
(Integral) apartment classes and the Ash–Rudolph theorem

Definition 2.9. A *frame* for F^n is a direct sum decomposition into lines

$$F^n = L_1 \oplus L_2 \oplus \cdots \oplus L_n$$

Given such a frame, let $S(L_1, L_2, \ldots, L_n)$ denote the simplicial subcomplex of $\mathcal{T}_n$ spanned by the vertices corresponding all direct sums of all nonempty proper subsets of the lines $\{L_1, L_2, \ldots, L_n\}$. The subcomplex $S(L_1, L_2, \ldots, L_n)$ is called an *apartment*.

The apartment $S(L_1, L_2, L_3)$ corresponding to a frame of F^3 is shown below.



Exercise 4. (a) Show that we can identify an apartment $S(L_1, L_2, \ldots, L_n)$, as a simplicial complex, with the barycentric subdivision of the boundary of an $(n-1)$-simplex. Conclude in particular that there is a homeomorphism $S(L_1, L_2, \ldots, L_n) \cong S^{n-2}$.

(b) Show that a choice of order on the lines $(L_1, L_2, \ldots, L_n)$ induces a choice of orientation on the sphere $S(L_1, L_2, \ldots, L_n) \cong S^{n-2}$. Deduce that for each choice of order (up to even permutations) we obtain a choice of fundamental class of $S(L_1, L_2, \ldots, L_n)$ in degree-$(n-2)$ homology with a well-defined sign.

Definition 2.10. Let $F^n = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ be a frame for F^n (ordered up to even permutations) and let $S(L_1, L_2, \ldots, L_n)$ be the associated apartment in $\mathcal{T}_n$. Then the image of the fundamental class of the sphere $S(L_1, L_2, \ldots, L_n)$ in $\mathrm{St}_n(F) = \widetilde{H}_{n-2}(\mathcal{T}_n(F))$, denoted $[S(L_1, L_2, \ldots, L_n)]$, is called an *apartment class*.

Theorem 2.11 (Solomon–Tits)**.** *The Steinberg module $\mathrm{St}_n(F)$ is generated by apartment classes for all $n \geq 2$.*

We do not expect, however, for the apartment classes to form a basis for $\mathrm{St}_n(F)$.

Definition 2.12. Let R be a domain and F its field of fractions. A frame $F^n = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ for F^n is called *integral* if

$$(L_1 \cap R^n) \oplus (L_2 \cap R^n) \oplus \cdots \oplus (L_n \cap R^n) = R^n.$$

Equivalently, if we chose a generator v_i for $(L_i \cap R^n)$ for all i, then the frame is integral if and only if the elements $\{v_1, \ldots, v_n\}$ form an R-basis for R^n. If $F^n = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ is an integral frame, then the apartment $S(L_1, L_2, \ldots, L_n)$ is called an *integral apartment* and $[S(L_1, L_2, \ldots, L_n)]$ an *integral apartment class*.

Theorem 2.13 (Ash–Rudolph [AR])**.** *Let R be a Euclidean domain and F its field of fractions. Then the Steinberg module $\mathrm{St}_n(F)$ is generated by integral apartment classes for all $n \geq 2$.*

A proof of Lee–Szczarba assuming Ash–Rudolph

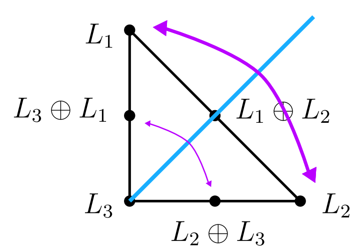
We can now prove that, if R is a Euclidean domain, then $H^{vcd_n}(\mathrm{SL}_n(R); \mathbb{Q})$ vanishes. Let F be the fraction field of R.

Recall that it suffices to show that the coinvariants $\mathrm{St}_n(F)_{\mathrm{SL}_n(R)}$ of the Steinberg module vanish. Since (by Ash–Rudolph's theorem) the Steinberg module is generated by integral apartment classes, it suffices to show that each integral apartment class is zero in the coinvariants. We do this in the following exercise.

Exercise 5. (a) Let $F^n = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ be an integral frame. Show there exists an element $g \in \mathrm{SL}_n(R)$ such that

$$\begin{aligned} g \cdot L_1 &= L_2 \\ g \cdot L_2 &= L_1 \\ g \cdot L_i &= L_i \text{ for all } i \geq 3. \end{aligned}$$

(b) Verify that the action of g on $\mathcal{T}_n(F)$ stabilizes and reverses the orientation on the apartment $S(L_1, L_2, \ldots, L_n)$. In the case $n = 3$, the reflection induced by g is illustrated below.



Deduce that

$$g \cdot [S(L_1, L_2, \ldots, L_n)] = -[S(L_1, L_2, \ldots, L_n)].$$

(c) Deduce that the apartment class $[S(L_1, L_2, \ldots, L_n)]$ vanishes in $\mathrm{SL}_n(R)$-coinvariants.

(d) Conclude that, when R is a Euclidean domain, $H^{vcd_n}(\mathrm{SL}_n(R); \mathbb{Q}) = 0$ for all $n \geq 2$.

3 Wilson Lecture 2: High connectivity of the partial basis complex implies the Ash–Rudolph theorem

Let's begin with a summary of the key points from the previous lecture.

Let R be the ring of integers in a number ring F.

- **Virtual Bieri–Eckmann duality**

$$H^{vcd_n - i}(\mathrm{SL}_n(R); \mathbb{Q}) \cong H_i(\mathrm{SL}_n(R); \mathbb{Q} \otimes \mathrm{St}_n(F))$$

with dualizing module the **Steinberg module**

$$\mathrm{St}_n(F) = \widetilde{H}_{n-2}(\mathcal{T}_n(F))$$

defined via the **Tits building** $\mathcal{T}_n(F)$

$$\begin{array}{rcl} \text{vertices} & \longleftrightarrow & \text{subspaces } 0 \subsetneq V \subsetneq F^n \\ p\text{-simplices} & \longleftrightarrow & \text{flags } 0 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_p \subsetneq F^n \end{array}$$

- **Solomon–Tits theorem**

$$\mathcal{T}_n(F) \simeq \bigvee S^{n-2}$$

$\mathrm{St}_n(F) = \widetilde{H}_{n-2}(\mathcal{T}_n(F))$ is generated by **apartment classes**

$$\begin{array}{rcl} \text{apartments} & \longleftrightarrow & \text{frames } F^n = L_1 \oplus L_2 \oplus \ldots \oplus L_n \\ S(L_1, L_2, \ldots, L_n) & = & \begin{array}{l}\text{full subcomplex of } \mathcal{T}_n(F) \text{ on vertices} \\ \text{corresponding to direct sums of the lines } L_i.\end{array} \\ & \simeq & S^{n-2} \end{array}$$

- **Ash–Rudolph theorem**

When R is Euclidean, $\mathrm{St}_n(F)$ is generated by **integral apartment classes**, i.e., $[S(L_1, L_2, \ldots, L_n)]$ such that $(L_1 \cap R^n) \oplus (L_2 \cap R^n) \oplus \cdots \oplus (L_n \cap R^n) = R^n$.

- **Lee–Szczarba theorem**

When R is Euclidean, $H^{vcd_n}(\mathrm{SL}_n(R); \mathbb{Q}) = 0$ for all $n \geq 2$.

Last Time: Ash–Rudolph $\implies$ Lee–Szczarba.

Today: Proof of Ash–Rudolph.

The goal for today's lecture is to prove Ash–Rudolph's theorem on the generation of the Steinberg module by integral apartment classes, assuming some intermediate results we will return to in the next lecture. We will give a simplified proof of Ash–Rudolph due to Church–Farb–Putman [CFP2].

Some methods in simplicial complexes

Simplicial complexes, simplices, and links

Recall that a simplicial complex X is the union of a set of simplices subject to the conditions that (1) every face of a simplex in X is a simplex, and (2) the intersection $\sigma_1 \cap \sigma_2$ of any two simplices in X is either empty or is a single face of both σ_1 and σ_2. These definitions are quite restrictive (compared to, say, the definition of a CW complex or even a Δ-complex) but they do mean that simplicial complexes can be neatly characterized combinatorially. Given a simplicial complex X, every p-simplex has $(p+1)$ distinct vertices. Moreover, every nonempty subset of vertices of X can span either zero or one simplex. Thus the complex X is completely determined by the data of its vertex set, and the collection of subsets of vertices that span simplices.

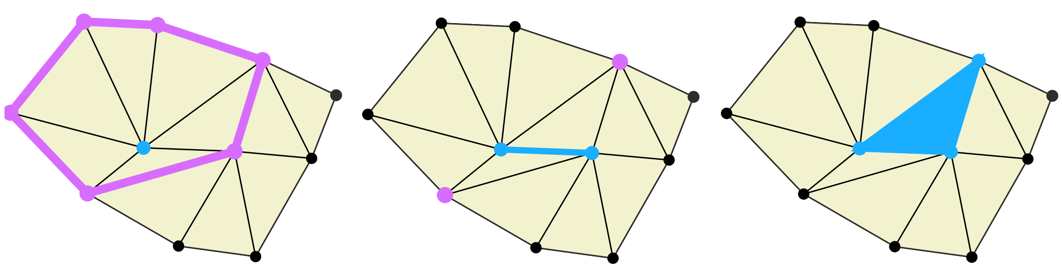
For a simplicial complex X, we write $\sigma = [s_0, s_1, \ldots, s_p]$ for the p-simplex spanned by the vertices $\{s_0, s_1, \ldots, s_p\}$. By abuse of notation, when convenient, we also use the notation $[s_0, s_1, \ldots, s_p]$ to encode an ordering on the vertices of the simplex, and understand the simplex σ to come with an orientation.

Definition 3.1. Let X be a simplicial complex, and $\sigma = [s_0, s_1, \ldots, s_p]$ a p-simplex in X. The *link* of σ in X is

$$\mathrm{Link}_X(\sigma) = \begin{array}{l}\text{subcomplex of } X \text{ consisting of the simplices} \\ \{[t_0, t_1, \ldots, t_q] \mid [s_0, s_1, \ldots, s_p, t_0, t_1, \ldots t_q] \text{ is a simplex in } X\}\end{array}$$

In other words, to find the link of σ, we consider all simplices containing σ as a face, and then take the union of all faces opposite σ.

Some simplices (blue) and their links (pink) are shown below. This image is modified from Wikipedia.



Connectivity

Recall the following convention from algebraic topology:

- all topological spaces are (-2)-*connnected*
- a space X is called (-1)-*connected* if and only if it is nonempty
- a space X is called *0-connected* if and only if it is path-connected
- a space X is called *1-connected* if and only if it is simply connected (i.e. path-connected with trivial fundamental group)
- in general, for $d \geq 0$, a space X is d-connected if and only if $\pi_i(X) = 0$ for all $0 \leq i \leq d$.

By the Hurewicz theorem, if a space X is d-connected, then $\widetilde{H}_i(X) = 0$ for all $-1 \leq i \leq d$.

The Cohen–Macaulay property

With the terminology above, we can make the following definition.

Definition 3.2. Let X be a d-dimensional simplicial complex. Then X is called *Cohen–Macaulay (CM)* if

- X is $(d-1)$-connected
- $\mathrm{Link}_X(\sigma)$ is $(d-2-\dim(\sigma))$-connected for all simplices σ in X.

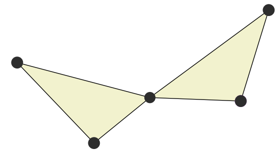
We may call X *CM of dimension* d to emphasize its dimension. By convention we say the empty set is CM of dimension -1.

This condition ensures that X is not only highly connected—at least as connected as a d-sphere—but has a 'nice' simplicial structure. For example, X might be the disk or sphere with a standard simplicial structure. This condition is often used in the topological stability literature to run inductive arguments, say, to prove that a family of simplicial complexes are increasingly highly connected.

Exercise 6. (a) Verify that the standard n-simplex is CM.

(b) Verify that the boundary of the standard n-simplex is CM. What about the k-skeleton?

(c) Verify that the following simplicial complexes is not CM, even though it is contractible. Conclude that high connectivity of a complex alone does not guarantee the CM property.



(d) Show by example that a subcomplex of a CM complex need not be CM.

A warning—there is an earlier, inequivalent, definition of CM in the literature. The original definition only gave a condition on the homology of the links, and not the homotopy groups. That version was a homeomorphism invariant. The version stated here, due to Quillen, is not; it depends on the simplicial structure.

Joins

The following concept and result will be useful in proving the CM property.

Definition 3.3. The *join* of simplicial complexes X and Y is a simplicial complex, denoted $X * Y$, defined as follows. Its vertex set is the disjoint union the vertices of X and Y. A subset of the vertex set spans a simplex if and only if it is the disjoint union of a (possibly empty) simplex in X and a (possibly empty) simplex in Y, viewed as subsets of their vertex sets.

Topologically, the join of spaces X and Y is homeomorphic to the quotient of $X \times Y \times [0,1]$ collapsing $X \times Y \times \{0\}$ to X and $X \times Y \times \{1\}$ to Y.

Exercise 7. Verify that $\mathrm{Link}_{\mathrm{Link}_X(\sigma)}(\tau) = \mathrm{Link}_X(\sigma * \tau)$.

See (for example) Milnor [Mi, Lemma 2.3] for a proof of the following.

Lemma 3.4. *Let X_0, X_1, $\ldots X_n$ be a collection of $(n+1)$ nonempty spaces. Then the join*

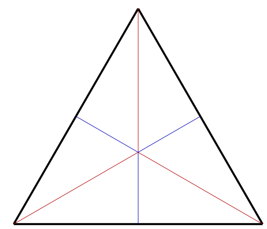
$$X_0 * X_1 * \cdots * X_n$$

is $(n-1)$-connected. More generally, if X_i is $(d_i - 1)$-connected for all i, then the join is $(d_0 + d_1 + \cdots + d_n + n - 1)$-connected.

Barycentric subdivision

Finally, we recall that the *barycentric subdivision* of a simplicial complex X (sometimes written $\mathrm{sd}(X)$) is the simplicial complex obtained from X by placing a new vertex in the centre of mass (the *barycentre*) of each simplex in X, and subdividing each simplex accordingly. A p-simplex in $\mathrm{sd}(X)$ corresponds to a flag of $(p+1)$-simplices (under inclusion) in X. This operation changes the simplicial structure but not the homeomorphism type of X.

The following image, showing the barycentric subdivision of a 2-simplex, was taken from Wikipedia.



The complex of partial bases and a theorem of Maazen

Fix a Euclidean domain R and let F be its field of fractions.

Definition 3.5. A set of vectors $v_0, \ldots, v_p$ in R^n is called a *partial basis* if it is a subset of a basis (or is a basis).

Exercise 8. (a) A vector $\{v_0\} \subseteq R^n$ is a partial basis if and only if v_0 is *primitive*, i.e., its entries generate R.

(b) The sets $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$ are not partial bases of $\mathbb{Z}^3$.

Definition 3.6. Fix a Euclidean ring R. We define an $(n-1)$-dimensional simplicial complex $PB_n(R)$, called the *complex of partial bases*, as follows.

- Vertices of $PB_n(R)$ are primitive vectors in R^n.
- A collection of vertices $\{v_0, \ldots, v_p\}$ span a p-simplex if and only if they are a partial basis.

We will sometimes consider instead the barycentric subdivision of $PB_n(R)$. This complex has a vertex for each partial basis of R^n, and simplices correspond to flags of partial bases. In other words, the barycentric subdivision is the simplicial complex associated to the poset of partial bases of R^n under inclusion.

Exercise 9. Choose a finite field R and sketch $PB_n(R)$ for some small values of n.

To prove Ash–Rudolph, we will use the following result of Maazen (see [Ma, Theorem 4.2]), which Maazen originally used to prove homological stability for the rings $\mathrm{GL}_n(R)$. We will outline a proof of this theorem in the next lecture.

Theorem 3.7 (Maazen [Ma])**.** *Let R be a Euclidean domain. The barycentric subdivision of $PB_n(R)$ is CM.*

Our goal is to relate the Tits building to $PB_n(R)$, in order to obtain a nice generating set for its homology. The key to this strategy is the following lemma of Quillen.

Quillen's lemma

Definition 3.8. Given a poset A, let $|A|$ denote the associated simplicial complex (called its *geometric realization*). The vertices of $|A|$ are elements of A, and its simplices correspond to flags.

Exercise 10. Let $\mathcal{A}$ be a simplicial complex. Let A be the poset of simplices of $\mathcal{A}$ under inclusion, and let $|A|$ be its geometric realization. Show that $|A|$ is the barycentric subdivision of $\mathcal{A}$. Conclude in particular that $|A|$ and $\mathcal{A}$ are homeomorphic as topological spaces.

Exercise 11. (a) Let A be a poset, and let $a \in A$. Show that the link of $\{a\}$ is the join

$$|A^{<a}| \quad * \quad |A^{>a}|$$

of the geometric realizations of the subposets

$$A^{<a} = \{a' \in A \mid a' < a\} \qquad \text{and} \qquad A^{>a} = \{a' \in A \mid a' > a\}.$$

(b) Given a chain $a_0 < a_1 < \cdots < a_p$ in A, what is the link of the simplex $[a_0, a_1, \ldots, a_p]$ in $|A|$?

Definition 3.9. For an element a in a poset A, the *height* $h(a)$ of a is the largest p such that there exists a chain $a_0 < a_1 < \ldots < a_p = a$ in A.

Lemma 3.10 (Quillen [Q, Theorem 9.1 and Corollary 9.7])**.** *Let $f : A \to B$ be a map of posets. Assume*

- *The map f is strictly increasing, i.e., $a < a' \implies f(a) < f(a')$.*
- *$|B|$ is CM of dimension d*
- *For all $b \in B$, the geometric realization $|f_b|$ of the "downward fibre"*

  $$f_b := \{a \in A \mid f(a) \leq b\} \subseteq A$$

  is CM of dimension $h(b)$. (Note f_b is sometimes denoted $f_{\leq b}$.)

Then $|A|$ is CM, and $f_* : \widetilde{H}_d(|A|) \to \widetilde{H}_d(|B|)$ surjects.

The proof of Ash–Rudolph

We now have the necessary ingredients to prove Ash–Rudolph's theorem on the generation of $\mathrm{St}_n(F)$ by integral apartment classes.

Let B be the poset of proper nonzero summands of F^n under inclusion, and A be the poset of *proper* partial bases of R^n under inclusion.

Exercise 12. Verify that the geometric realization $|B|$ is the Tits building $\mathcal{T}_n(F)$, and $|A|$ is the barycentric subdivision of the $(n-2)$-skeleton of the partial bases complex $PB_n(R)$.

Consider the map of posets

$$\begin{aligned} f : A &\longrightarrow B \\ \{v_0, \dots v_p\} &\longmapsto \mathrm{span}_F\{v_0, \dots v_p\} \end{aligned}$$

In the following exercise, we will check that the hypotheses of Quillen's Lemma (Lemma 3.10) holds for f.

Exercise 13. (a) Verify that f is strictly increasing.

(b) Verify that $|B| = \mathcal{T}_n(F)$ is CM of dimension $(n-2)$.
Hint: Use Exercise 11, Lemma 3.4, and the Solomon–Tits theorem. Relate $B^{<V}$ to the Tits building on the F-vector space V, and $B^{>V}$ to the Tits building on the F-vector space F^n/V. What can you say about the subposet $\{W \in B \mid U \subsetneq W \subsetneq V\}$ for fixed U, V?

(c) For all $V \in F^n$, verify that f_V is the complex of partial bases on V. Conclude that $|f_V|$ is CM of the appropriate dimension.

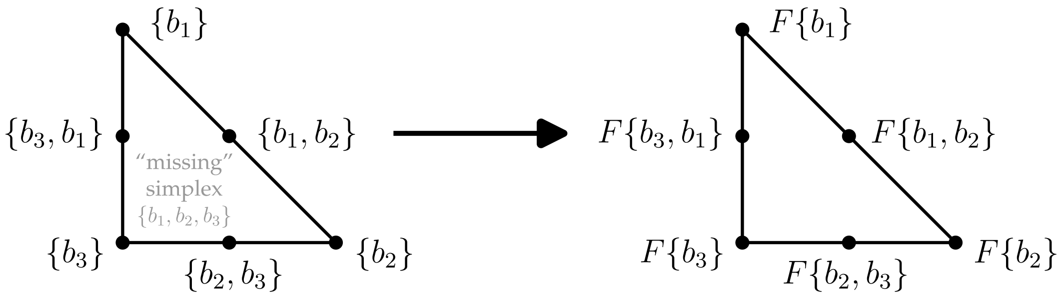
It then follows from Lemma 3.10 that we have a surjection

$$f_* : \widetilde{H}_{n-2}(|A|) \longrightarrow \widetilde{H}_{n-2}(\mathcal{T}_n(F)) = \mathrm{St}_n(F).$$

To conclude the proof, we will show that the integral apartment classes are the image of a generating set for $\widetilde{H}_{n-2}(|A|)$.

Since $PB_n(R)$ is $(n-2)$-connected, we might expect that the degree-$(n-2)$ homology of its $(n-2)$-skeleton be generated by the boundaries of the $(n-1)$-simplices "missing" from the $(n-2)$-skeleton. We will confirm this intuition using the long exact sequence of a pair.

Once we do this, we are done: the $(n-1)$-simplices in $PB_n(R)$ correspond to bases of R^n, and the images of their (subdivided) boundaries are precisely the *integral* apartments in $\mathcal{T}_n(F)$, as sketched below. Then, using the Quillen Lemma, we conclude that integral apartment classes generate $\mathrm{St}_n(F) = \widetilde{H}_{n-2}(\mathcal{T}_n(F))$.



So consider the pair $(PB_n(R), |A|)$. From the associated long exact sequence, we have an exact sequence

$$H_{n-1}(PB_n(R), |A|) \xrightarrow{(*)} \widetilde{H}_{n-2}(|A|) \longrightarrow \widetilde{H}_{n-2}(PB_n(R))$$

By Maazen's high-connectivity result for $PB_n(R)$ (Theorem 3.7), the third term vanishes, so the connecting homomophism $(*)$ must surject, and we have a surjective map

$$H_{n-1}(PB_n(R), |A|) \twoheadrightarrow \widetilde{H}_{n-2}(|A|) \overset{f_*}{\twoheadrightarrow} \widetilde{H}_{n-2}(\mathcal{T}_n(F)) = \mathrm{St}_n(F).$$

But (using the original simplicial structure on $PB_n(R)$, viewing the topological space $|A|$ as its codimension-1 skeleton, and working with simplicial homology) we can identify the relative homology group $H_{n-1}(PB_n(R), |A|)$ with the group of simplicial $(n-1)$-chains on $PB_n(R)$, in other words, the free abelian group on bases for R^n. The connecting homomorphism takes an $(n-1)$-chain to its boundary. We conclude that $\mathrm{St}_n(F)$ is generated by integral apartment classes, and complete the proof.

Summary of today's results

Recall R is a Euclidean domain and F its field of fractions.

- We proved **Ash–Rudolph's** theorem, which states that when R is Euclidean, the Steinerg module $\mathrm{St}_n(F)$ is generated by **integral apartment classes**, i.e., $[S(L_1, L_2, \ldots, L_n)]$ where the frame $L_1, L_2, \ldots, L_n$ arises from a basis of R^n (not just F^n).

- We proved the result by constructing a surjection $\widetilde{H}_{n-2}(|A|) \to \mathrm{St}_n(F)$, where the space $|A|$ is the $(n-2)$-skeleton of the **complex of partial bases** $PB_n(R)$.

- We proved this map surjects using a lemma of **Quillen** on maps of posets. To apply the lemma we used the **Solomon–Tits** theorem (the Tits building $\mathcal{T}_n(F)$ is CM of dimension $(n-2)$) and a theorem of **Maazen** (the complex of partial bases $PB_n(R)$ is CM of dimension $(n-1)$).
- We used Maazen's theorem a second time to argue that the (surjective) image of the map $\widetilde{H}_{n-2}(|A|) \to \mathrm{St}_n(F)$ is generated by integral apartment classes.

Tomorrow: We will give a proof of Maazen's result. This will complete the argument

$$\text{Maazen} \quad \Longrightarrow \quad \text{Ash–Rudolph} \quad \Longrightarrow \quad \text{Lee–Szczarba,}$$

where Lee–Szczarba states $H^{vcd_n}(\mathrm{SL}_n(R); \mathbb{Q}) = 0$ for R Euclidean and all $n \geq 2$.

4 Wilson Lecture 3: The complex of partial bases is highly connected

Let's begin with a summary of the key points from the previous lecture.

Let R be the ring of integers in a number field F.

- **Virtual Bieri–Eckmann duality**

$$H^{vcd_n - i}(\mathrm{SL}_n(R); \mathbb{Q}) \cong H_i(\mathrm{SL}_n(R); \mathbb{Q} \otimes \mathrm{St}_n(F))$$

  with dualizing module the **Steinberg module**

$$\mathrm{St}_n(F) = \widetilde{H}_{n-2}(\mathcal{T}_n(F))$$

  defined via the **Tits building** $\mathcal{T}_n(F)$

$$\begin{aligned} \text{vertices} \quad &\longleftrightarrow \quad \text{subspaces } 0 \subsetneq V \subsetneq F^n \\ p\text{-simplices} \quad &\longleftrightarrow \quad \text{flags } 0 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_p \subsetneq F^n \end{aligned}$$

- **Solomon–Tits theorem**

$$\mathcal{T}_n(F) \simeq \bigvee S^{n-2}$$

$\mathrm{St}_n(F) = \widetilde{H}_{n-2}(\mathcal{T}_n(F))$ is generated by **apartment classes**

$$\begin{array}{rcl} \text{apartments} & \longleftrightarrow & \text{frames } F^n = L_1 \oplus L_2 \oplus \ldots \oplus L_n \\ S(L_1, L_2, \ldots, L_n) & = & \begin{array}{l}\text{full subcomplex of } \mathcal{T}_n(F) \text{ on vertices} \\ \text{corresponding to direct sums of the lines } L_i.\end{array} \\ & \simeq & S^{n-2} \end{array}$$

- **Church–Farb–Putman conjectures**

  Conjecturally, $H^{vcd_n - i}(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q}) = 0$ for all $n \geq i + 2$.

- **Lee–Szczarba theorem**

  When R is Euclidean, $H^{vcd_n}(\mathrm{SL}_n(R); \mathbb{Q}) = 0$ for all $n \geq 2$.

- **Ash–Rudolph theorem**

  When R is Euclidean, $\mathrm{St}_n(F)$ is generated by **integral apartment classes**, i.e., $[S(L_1, L_2, \ldots, L_n)]$ such that $(L_1 \cap R^n) \oplus (L_2 \cap R^n) \oplus \cdots \oplus (L_n \cap R^n) = R^n$.

- The **the complex of partial bases** $PB_n(R)$

$$\begin{array}{rcl} \text{vertices} & \longleftrightarrow & \text{primitive vectors } v \in R^n \\ p\text{-simplices} & \longleftrightarrow & \text{partial bases } \{v_0, v_1, \ldots, v_p\} \text{ for } R^n \end{array}$$

  Maazen theorem

  When R is Euclidean, $\mathrm{sd}(PB_n(R))$ is CM of dimension $(n-1)$.
  i.e., $PB_n(R)$ and its links are as highly connected as a standard $(n-1)$-sphere

Wilson Lecture 1: Ash–Rudolph $\Longrightarrow$ Lee–Szczarba.

Wilson Lecture 2: Maazen $\Longrightarrow$ Ash–Rudolph.

Today: Proof of Maazen

The goal for today's lecture is to prove Maazen's result that the complex of partial bases $PB_n(R)$ is CM of dimension $(n-1)$ when R is Euclidean. This will complete our proof of Ash–Rudolph's theorem, and, in particular, the codimension-0 case of the Church–Farb–Putman vanishing conjecture on the high-degree cohomology of $\mathrm{SL}_n(\mathbb{Z})$.

We will use a proof due to Church–Putman [CP]. This approach will showcase some of the methods (in a warm-up setting) used by Church–Putman [CP] and Brück–Miller–Patzt–Sroka–Wilson [BMPSW] to prove the codimension-1 and codimension-2 case of the Church–Farb–Putman conjectures.

For simplicity we will specialize to the case $R = \mathbb{Z}$, though emphasize when we use its Euclidean property. It is left as an exercise to the reader to adapt the arguments to general Euclidean rings.

Exercise 14. Explain how to generalize the following proof from $R = \mathbb{Z}$ to a arbitrary Euclidean ring R. Verify that the arguments hold in this generality.

Today's goal and proof outline

Let $R = \mathbb{Z}$. Write PB_n for $PB_n(\mathbb{Z})$.

Definition 4.1. Let $e_1, \ldots, e_k$ be the standard basis for $\mathbb{Z}^k$. Following Church–Putman, we adopt the notation

$$PB_n^m := \mathrm{Link}_{PBn+m}(\{e_1, e_2, \ldots, e_m\}).$$

By convention $PB_n^0 = PB_n$, the link of the empty simplex.

This definition says that PB_n^m is the subcomplex spanned of PB_{m+n} of simplices $\{v_1, \ldots v_p\}$ such that $\{e_1, e_2, \ldots, e_m, v_1, \ldots, v_p\}$ is a partial basis for $\mathbb{Z}^{m+n}$. In other words, it is the complex of partial bases of direct complements to $\mathbb{Z}^m = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_m \subseteq \mathbb{Z}^{m+n}$.

Exercise 15. Verify the following.

(a) PB_0^m is empty.

(b) The complex PB_n^m has dimension $(n-1)$ for all $m \geq 0$.
Use the convention that the empty set has dimension -1.

Our goal for this lecture is to prove the following theorem.

Theorem 4.2. *For all $m, n \geq 0$, PB_n^m is CM of dimension $(n-1)$.*

When $m = 0$, this is Maazen's result, our stated goal for today's lecture.

Proof strategy

Our goal is to prove that PB_n^m is CM of dimension $(n-1)$ for all $m, n \geq 0$. This means we must check, for all $m, n \geq 0$,

- PB_n^m is $(n-1)$-dimensional. This is Exercise 15.
- PB_n^m is $(n-2)$-connected.
- For any simplex $\sigma \in PB_n^m$ the link $\mathrm{Link}_{PB_n^m}(\sigma)$ is $(n - \dim(\sigma) - 3)$-connected.

We will use the following approach.

- We proceed by induction on n. For each $n \geq 0$, we assume $PB_{n'}^{m'}$ is CM of dimension $(n'-1)$ for all $m' \geq 0$ and $n' < n$.
- Fix m and consider PB_n^m. We will identify the links of (nonempty) simplices in PB_n^m with complexes $PB_{n'}^{m'}$ for $n' < n$. Thus the connectivity of the links follows from the induction hypothesis. Our goal is to prove PB_n^m is $(n-2)$-connected, that is, $\pi_p(PB_n^m) = 0$ for $0 \leq p \leq (n-2)$.
- Fix $0 \leq p \leq (n-2)$ and consider a continuous map

  $$\phi : S^p \longrightarrow PB_n^m$$

  from a p-sphere. We wish to show ϕ is nullhomotopic, so $\pi_p(PB_n^m) = 0$. We can assume (by the Simplicial Approximation Theorem) that ϕ is a simplicial map with respect to some simplicial structure on S^p. Since S^p is compact, this structure has finitely many simplices.
- We will define a function $R = R_\phi$ on the vertices x of S^p to $\mathbb{Z}_{\geq 0}$ that we view as a measure of the 'badness' of the image $\phi(x)$. This function R is defined so that, if we can homotope ϕ to reduce R to zero at every vertex, the resultant map will be demonstrably nullhomotopic.

  Our goal becomes the following: Let $N = \max_{x \in S^p} R(x)$. We will show that we can homotope ϕ to reduce the R-value of (at least one) vertex in S^p with R-value N, without creating any new vertices of R-value equal to N or greater. We are then done by induction. This style of high-connectivity argument is sometimes called a "badness argument".
- To remove 'bad' vertices from $\phi(S^p)$ we homotope the map ϕ to "push" the image $\phi(x)$ into its link. To ensure we can do this without introducing new 'bad' vertices, we must study the subcomplexes of the links in PB_n^m with small badness-values.

Links in PB_n^m

The following lemma, which describes links in PB_n^m, is left as an exercise.

Lemma 4.3. *Given a simplex σ in PB_m^n, there is an isomorphism of simplicial complexes*

$$\mathit{Link}_{PB_n^m}(\sigma) \cong PB_{n-\mathit{dim}(\sigma)-1}^{m+\mathit{dim}(\sigma)+1}.$$

Exercise 16. Verify that $\mathrm{Link}_{PB_n^m}(\sigma) \cong PB_{n-\dim(\sigma)-1}^{m+\dim(\sigma)+1}$.
Hint: Let $\sigma = \{v_1, \dots v_q\}$. Verify that a "change of basis" transformation of $\mathbb{Z}^{n+m}$ induces a simplicial automorphism of PB_{n+m}.

Beginning the proof of Theorem 4.2

Proof of Theorem 4.2. We will prove PB_n^m is CM of dimension $(n-1)$ for all $m, n \geq 0$. We proceed by induction on n.

The base case was addressed in Exercise 15: when $n = 0$, PB_0^m is the link of a maximal-dimensional simplex and therefore is empty. Thus PB_0^m is CM of dimension -1 for all m.

Fix $n \geq 0$. Assume by induction that $PB_{n'}^{m'}$ is CM of dimension $(n'-1)$ for all $0 \leq n' < n$ and all $m' \geq 0$. Our goal is to prove PB_n^m is CM of dimension $(n-1)$. We already know

- PB_n^m is dimension $(n-1)$ (Exercise 15).
- For all simplices $\sigma \in PB_n^m$,

$$\mathrm{Link}_{PB_n^m}(\sigma) \cong PB_{n-\dim(\sigma)-1}^{m+\dim(\sigma)+1}$$

  by Lemma 4.3. This complex is CM of dimension $(n - \dim(\sigma) - 2)$ by the inductive hypothesis. In particular it is $(n - \dim(\sigma) - 3)$-connected, as required.

It remains to show that PB_n^m is $(n-2)$-connected.

Fix $0 \leq p \leq (n-2)$, and let

$$\phi : S^p \longrightarrow PB_n^m$$

be a map from a p-sphere. After possibly modifying ϕ up to homotopy, we can assume this map is simplicial with respect to some simplicial structure on S^p. In fact, it is a (not entirely trivial) result from PL topology that we can assume that S^p is a *combinatorial p-sphere*; the links of its simplices are simplicial spheres. Our goal is to show that ϕ is nullhomotopic. We will proceed by a 'badness argument' as outlined in the proof summary above.

Define the following function on the vertices of PB_n^m. Recall that a vertex is a primitive element in $\mathbb{Z}^{m+n}$, which we may view as an $(m+n)$-vector.

$$F : \{\text{vertices of } PB_n^m\} \longrightarrow \mathbb{Z}_{\geq 0}$$
$$v \longmapsto |(m+n)\text{th coordinate of } v|$$

Now we can define, associated to the map ϕ, our 'badness' function $R = R_\phi$

$$R_\phi : \{\text{vertices of } S^p\} \longrightarrow \mathbb{Z}_{\geq 0}$$
$$x \longmapsto F(\phi(x))$$

Let $R^{\max} = \max_{x \in S^p} R(x)$. Our goal is to homotope ϕ so that $R^{\max} = 0$. The following exercise shows that the resulting map is nullhomotopic.

Exercise 17. Suppose that $R_\phi(x) = 0$ for all $x \in S^p$.

(a) Show that the image of ϕ is in the link of vertex e_{n+m} of PB_n^m.

(b) Conclude that we can homotope ϕ to the constant map at the point e_{n+m}.

We will show that, if $R^{\max} = N > 0$, then we can homotope ϕ to reduce $R(x)$ for some vertex x with $R(x) = N$, without raising the R-value of any other vertices to N. This will complete the proof.

Assume $R^{\max} = N > 0$. Let τ be a simplex in S^p of maximal dimension having the property that $R(x) = N$ for all $x \in \tau$. There could be more than one such simplex; it is enough to show we can resolve one of them. Our strategy is to homotope ϕ to 'push' the image of τ off the simplex $\phi(\tau)$ and into the link of $\phi(\tau)$. To ensure we can do this in a way that reduces R-values of τ, we need to study the subcomplex of the links of simplices in PB_n^m of small F-values.

The subcomplexes $\mathrm{Link}_{PB_n^m}(\sigma)^{<N}$

Recall that F is the 'absolute value of the last coordinate' function on vectors in $\mathbb{Z}^{m+n}$.

Definition 4.4. Fix $N \geq 0$ in $\mathbb{Z}$. For a subcomplex X of PB_n^m, we use the notation $X^{<N}$ to denote the subcomplex of X spanned by vertices $v \in X$ satisfying $F(v) < N$.

Lemma 4.5. *Let σ be a simplex in PB_n^m, and suppose w is a vertex of σ such that $F(w) = N > 0$. Then there exists a retraction $\pi = \pi_{\sigma,w}$*

$$\pi : \mathit{Link}_{PB_n^m}(\sigma) \longrightarrow \mathit{Link}_{PB_n^m}(\sigma)^{<N}.$$

We will construct π as a simplicial map. This means we will first define it on vertices. Then, to show it is a well-defined, continuous map, we only need to check that for every collection of vertices in the domain $\mathrm{Link}_{PB_n^m}(\sigma)$ that span a simplex, their image under π span a simplex.

Define

$$\begin{aligned} \pi : \mathrm{Link}_{PB_n^m}(\sigma) &\longrightarrow \mathrm{Link}_{PB_n^m}(\sigma)^{<N} \\ v &\longmapsto v - qw \end{aligned}$$

where $q \in \mathbb{Z}$ is determined by the Euclidean algorithm: it is the quotient of the last coordinate of v on division by the last coordinate of w.

For example, suppose $w = \begin{bmatrix} 6 \\ 1 \\ 10 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 5 \\ 43 \end{bmatrix}$. When we apply the Euclidean algorithm to the last coordinates, we find $43 = 4(10) + 3$, with quotient $q = 4$. Then

$$\pi(v) = v - 4w = \begin{bmatrix} 2 \\ 5 \\ 43 \end{bmatrix} - 4 \begin{bmatrix} 6 \\ 1 \\ 10 \end{bmatrix} = \begin{bmatrix} -22 \\ 1 \\ 3 \end{bmatrix}.$$

In this example, the remainder $3 < R = 10$ as desired.

We will verify that π is the desired retraction in the following exercise.

Exercise 18. (a) Suppose that $\{e_1, \ldots, e_m, w_1, \ldots, w_k, v_1, \ldots, v_\ell\}$ is a partial basis for $\mathbb{Z}^{m+n}$. Explain why, for any $w \in \{w_1, \ldots, w_k\}$ and integers $q_1, \ldots, q_\ell$, the set

$$\{e_1, \ldots, e_m, w_1, \ldots, w_k, v_1 - q_1 w, \ldots, v_\ell - q_\ell w\}$$

is a partial basis for $\mathbb{Z}^{m+n}$. In particular the elements $v_i - q_i w$ are primitive.

(b) Deduce that $\pi(v)$ is an element of $\mathrm{Link}_{PB_n^m}(\sigma)$ for all $v \in \mathrm{Link}_{PB_n^m}(\sigma)$.

(c) Explain why in fact $\pi(v) \in \mathrm{Link}_{PB_n^m}(\sigma)^{<N}$. Conclude that π is well-defined on vertices.

(d) Again use part (a) to deduce that π extends over simplices. Concretely: if $\{v_1, \ldots, v_\ell\}$ spans a simplex in $\mathrm{Link}_{PB_n^m}(\sigma)$, show that $\{\pi(v_1), \ldots, \pi(v_\ell)\}$ spans a simplex in $\mathrm{Link}_{PB_n^m}(\sigma)^{<N}$. Conclude that π is a well-defined simplicial map.

(e) Verify that π fixes $\mathrm{Link}_{PB_n^m}(\sigma)^{<N}$ pointwise. Conclude that π is a retraction, as claimed.

The key to executing our 'badness' argument is the following result.

Corollary 4.6. *Let σ be a simplex in PB_n^m. Assume $\mathrm{Link}_{PB_n^m}(\sigma)$ is $(n - \dim(\sigma) - 3)$-connected (as implied by our inductive hypothesis in the proof of Theorem 4.2). Then $\mathrm{Link}_{PB_n^m}(\sigma)^{<R}$ is also $(n - \dim(\sigma) - 3)$-connected.*

We prove this corollary in the following exercise.

Exercise 19. (a) Suppose that $\pi : X \to Y$ is a retraction of topological spaces. Explain why π induces surjective maps on homotopy groups.

(b) Deduce Corollary 4.6 from Lemma 4.5. That is, show that if $\mathrm{Link}_{PB_n^m}(\sigma)$ is $(n - \dim(\sigma) - 3)$-connected, then so is $\mathrm{Link}_{PB_n^m}(\sigma)^{<R}$, using the existence of a retraction

$$\mathrm{Link}_{PB_n^m}(\sigma) \to \mathrm{Link}_{PB_n^m}(\sigma)^{<R}.$$

Resuming the proof of Theorem 4.2

Recall that we had a map $\phi : S^p \to PB_n^m$ with associated badness measure $R(x) = F(\phi(x))$ on vertices of S^p, and total badness $R^{\max} = \max_{x \in S^p} R(x) = N > 0$. Our goal is to homotope ϕ to reduce $R^{\max}$. Call a simplex in S^p *bad* if every vertex has maximally-bad R-value N. We had selected a maximal-dimensional bad simplex τ in S^p. To complete the proof, we will homotope ϕ to reduce the R-values of the vertices of τ, without increasing the R-values of any vertices to N.

Suppose that $\dim(\tau) = k$. Let $\dim(\phi(\tau)) = \ell$. The map $\phi|_\tau$ may or may not be injective, but we know $\ell \leq k$.

By construction of our simplicial structure on S^p, we know $\mathrm{Link}_{S^p}(\tau) \cong S^{p-k-1}$. By our assumption that τ was maximal-dimensional, we know $R(x) < N$ for any $x \in \mathrm{Link}_{S^p}(\tau)$, which implies $\phi(x) \notin \phi(\tau)$. Thus

$$\phi(\mathrm{Link}_{S^p}(\tau)) \subseteq \mathrm{Link}_{PB_n^m}(\phi(\tau)).$$

In fact, since $R(x) < N$, we know

$$\phi(\mathrm{Link}_{S^p}(\tau)) \subseteq \mathrm{Link}_{PB_n^m}(\phi(\tau))^{<N}.$$

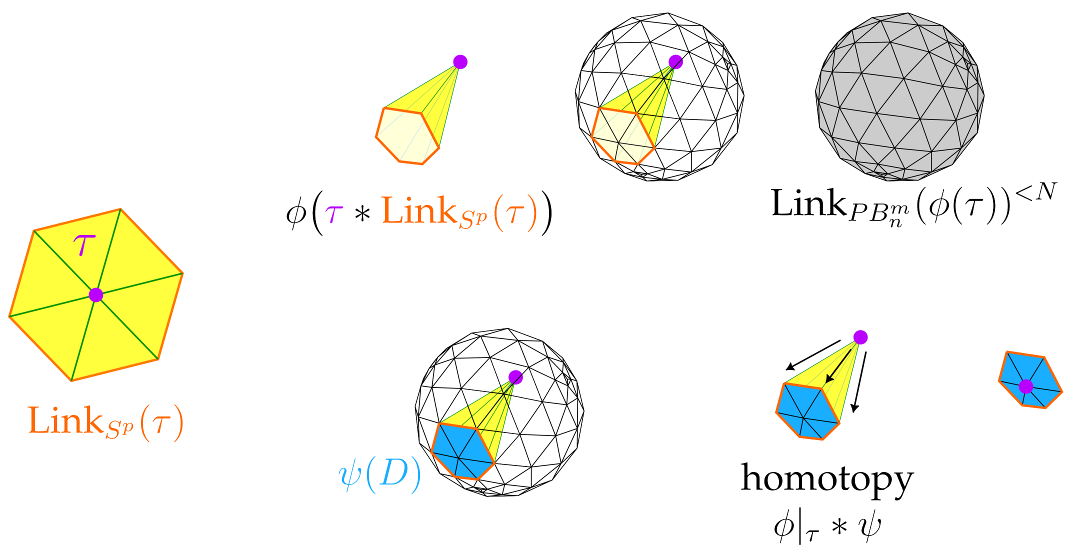
But $\mathrm{Link}_{S^p}(\tau) \cong S^{p-k-1}$ and (by Corollary 4.6) our inductive hypothesis implies that $\mathrm{Link}_{PB_n^m}(\phi(\tau))^{<N}$ is $(n-\ell-3)$-connected. By the following exercise, then, the image of the sphere $\mathrm{Link}_{S^p}(\tau)$ in $\mathrm{Link}_{PB_n^m}(\phi(\tau))^{<N}$ is nullhomotopic.

Exercise 20. Verify that $p - k - 1 \leq n - \ell - 3$. Conclude that $\phi|_{\mathrm{Link}_{S^p}(\tau)}$ is nullhomotopic in $\mathrm{Link}_{PB_n^m}(\phi(\tau))^{<N}$.

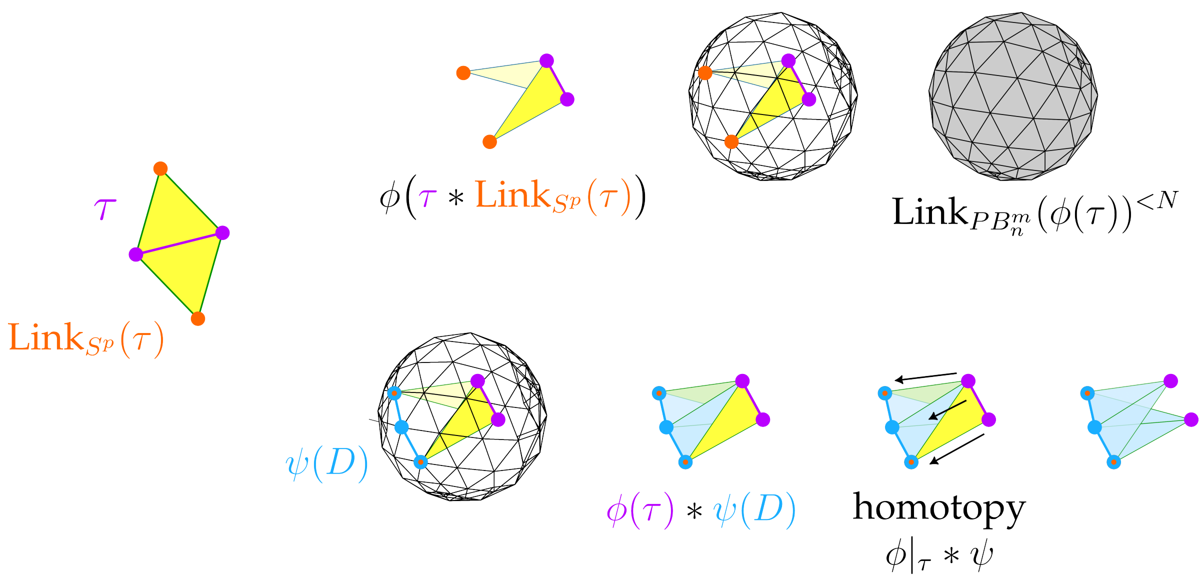
This means that the image of the sphere $\mathrm{Link}_{S^p}(\tau)$ bounds a $(p-k)$-disk in $\mathrm{Link}_{PB_n^m}(\phi(\tau))^{<N}$. In fact, using more results from PL topology, we can deduce that there exists a *combinatorial $(p-k)$-disk D*—a simplicial disk whose links are spheres—with $\partial D \cong \mathrm{Link}_{S^p}(\tau)$, and a simplicial map $\psi : D \to \mathrm{Link}_{PB_n^m}(\phi(\tau))^{<N}$ that restricts to $\phi|_{\mathrm{Link}_{S^p}(\tau)}$ on the boundary of D. See Church–Putman [CP, Proof of Theorem 4.2] for more PL topology details.

We claim that we can homotope ϕ to replace ϕ on the p-disk $\tau * \mathrm{Link}_{S^p}(\tau) \subseteq S^p$ with a new map defined using ψ, while fixing ϕ on on the boundary and the complement of the disk $\tau * \mathrm{Link}_{S^p}(\tau)$. The homotopy will be continuous but may not be simplicial. We typically need to alter the simplicial structure on S^p to make the new map simplicial, but this is acceptable as long we do not introduce new maximal-dimensional bad simplices.

The following figures illustrate this procedure. The first figure shows a case in which τ is a vertex. In this sequence, the image of $\tau * \mathrm{Link}_{S^p}(\tau)$ (shaded yellow) is replaced (via the homotopy) by the disk shaded blue. The formerly-bad vertex τ now maps to a vertex in $\mathrm{Link}_{PB_n^m}(\phi(\tau))^{<N}$.



The next figure shows a case in which τ is an edge. Note in this case the homotopy does not change the image of any vertex. The image of $\tau * \mathrm{Link}_{S^p}(\tau)$ (shown in yellow) is replaced (via our homotopy) by the four triangles shaded blue shown in final panel. However, we have modified the simplicial structure on $\tau * \mathrm{Link}_{S^p}(\tau)$ so that the two bad simplices $\partial\tau$ are no longer joined by an edge.



We can formalize this argument as follows. Consider the map from the $(p+1)$-ball

$$\phi|_\tau * \psi : \tau * D \longrightarrow PB_n^m.$$

The boundary of this $(p+1)$-ball is a p-sphere that we can write as the union

$$\partial(\tau * D) = (\tau * \partial D) \cup_{\partial\tau * \partial D} (\partial\tau * D).$$

The map $(\phi|_\tau * \psi)$ restricts on $(\tau * \partial D) \cong (\tau * \mathrm{Link}_{S^p}(\tau))$ to $\phi|_{\tau * \mathrm{Link}_{S^p}(\tau)}$.

Thus the map $(\phi|_\tau * \psi)$ defines a homotopy from $\phi|_{\tau * \mathrm{Link}_{S^p}(\tau)}$ to $(\phi|_{\partial\tau} * \psi)$.

This homotopy fixes the values of ϕ on

$$\partial(\tau * \mathrm{Link}_{S^p}(\tau)) = \partial\tau * \mathrm{Link}_{S^p}(\tau) \cong \partial\tau * \partial D$$

and so we can extend it to fix ϕ on the rest of S^p pointwise.

Exercise 21. Verify that a simplex τ has the same vertex set as its boundary $\partial\tau$ unless τ is a point, in which case $\partial\tau = \varnothing$.

At this point S^p may have the same number of bad vertices, but (on the subcomplex $\tau * \mathrm{Link}_{S^p}(\tau)$ where we performed the homotopy) the maximal-dimensional bad simplices now have dimension strictly less than τ did. Specifically, since ψ maps every vertex of D to $\mathrm{Link}_{PB_n^m}(\phi(\tau))^{<N}$, the maximal-dimensional bad simplices in $\partial\tau * D$ are the faces of τ. If $\dim(\tau) = 0$, then $\partial\tau = \varnothing$ and the procedure modifies the map ϕ to have a strictly smaller

R-value on the vertex τ.

Thus, by iterating this procedure, we can eventually homotope ϕ to have R-value strictly less than N on every vertex in the disk $\tau * \mathrm{Link}_{S^p}(\tau)$. Repeating this procedure for other maximal-dimensional bad simplex will lower $R^{\max}$, and by induction we can reduce $R^{\max}$ to 0 and conclude the proof.

Exercise 22. Church–Putman did not actually prove this result for the complex $PB_n(R)$ of partial bases. Instead, they proved it for a closely related complex, the *complex of partial frames* $B_n(R)$. Vertices of $B_n(R)$ are lines in R^n. It may be convenient to write a line as an equivalence class of primtive vectors v, defined up to multiplication by a unit. A set of lines span a simplex if they are a subset of a frame, equivalently, if their associated primitive vectors are a partial basis. Adapt the arguments from this lecture to prove that $B_n(R)$ is CM of dimension $(n-1)$.

5 Patzt Lecture 2: Codimension 1 cohomology of $\mathrm{SL}_n\mathbb{Z}$

In this lecture, we want to understand the structure of Church–Putman's proof that

$$H^{\binom{n}{2}-1}(\mathrm{SL}_n\mathbb{Z};\mathbb{Q}) \cong 0 \quad \text{for } n \geq 3.$$

This result can be used to prove that $K_8(\mathbb{Z}) \cong 0$, a recent result by Dutour Sikirić–Elbaz-Vincent–Kupers–Martinet.

We will start with a presentation of the Steinberg module, then argue how this presentation implies the result. Afterwards, we give a sketch of how to derive the presentation.

The Bykovskii presentation: As we have seen in Jenny's lectures, $\mathrm{St}_n\mathbb{Q}$ is generated by integral apartments classes, which are $[S(L_1,\dots,L_n)] \in \widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q}))$ for a line decomposition $L_1 \oplus \cdots \oplus L_n = \mathbb{Z}^n$. Some relations between such apartment classes are easy to see and have been covered in the exercises:

$$[S(L_1,\dots,L_n)] = (-1)^\pi \cdot [S(L_{\pi(1)},\dots,L_{\pi(n)})] \quad \text{for all permutations } \pi \in S_n.$$

There is one more relation that is less obvious—the Manin relation:

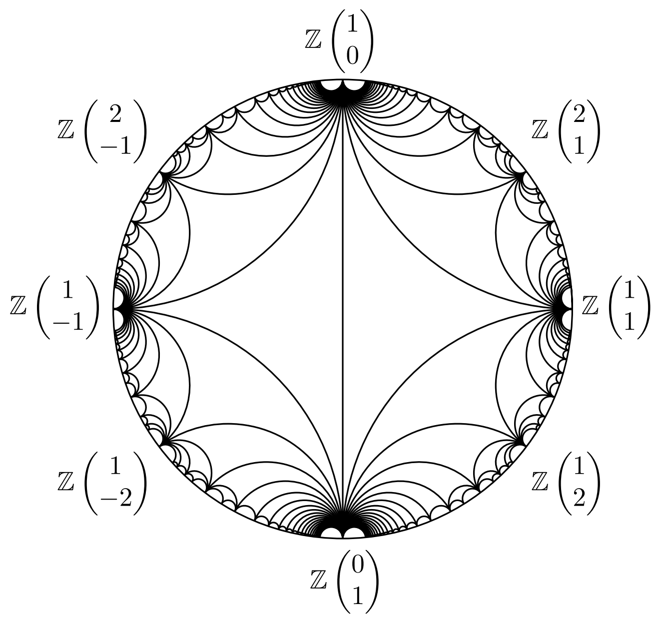
$$[S(L_1,L_2,L_3,\dots,L_n)] = [S(L_0,L_2,L_3,\dots,L_n)] + [S(L_1,L_0,L_3,\dots,L_n)],$$

where L_0 is generated by v_1+v_2 with $L_1 = \mathbb{Z}v_1$ and $L_2 = \mathbb{Z}v_2$.

We will describe this presentation in another way.

Some simplicial complexes: Recall that $B_n(\mathbb{Z})$ is the complex of partial frames. An *augmented frame* is a set of lines $\{L_0,\dots,L_n\}$ such that (after reordering) $L_1 \oplus \cdots \oplus L_n = \mathbb{Z}^n$ (i.e. is a frame) and L_0 is generated by v_1+v_2 with $L_1 = \mathbb{Z}v_1$ and $L_2 = \mathbb{Z}v_2$.

$BA_n(\mathbb{Z})$ is the simplicial complex whose vertices are lines in $\mathbb{Z}^n$ and whose simplices are subsets of augmented frames. Here is a picture for $n = 2$.



Note that $\mathrm{SL}_n\mathbb{Z}$ acts on $BA_n(\mathbb{Z})$ and thus $C_n(BA_n(\mathbb{Z}))$ is a $\mathrm{SL}_n\mathbb{Z}$-module.

Exercise 23. (hard) Show that $C_n(BA_n(\mathbb{Z}); \mathbb{Q})$ and $C_{n-1}(B_n(\mathbb{Z}); \mathbb{Q})$ are projective $\mathrm{SL}_n\mathbb{Z}$-modules. (Hint: Consider the stabilizer subgroup of an n-simplex in $BA_n(\mathbb{Z})$.)

We will see a little later that

$$\begin{aligned} C_n(BA_n(\mathbb{Z})) &\longrightarrow C_{n-1}(B_n(\mathbb{Z})) \\ [L_0, \ldots, L_n] &\longmapsto [L_1, \ldots, L_n] - [L_0, L_2, \ldots, L_n] + [L_0, L_1, L_3, \ldots, L_n] \end{aligned}$$

is $\mathrm{SL}_n\mathbb{Z}$-equivariant.

The Bykovskii presentation rephrased: By sending $[L_1, \ldots, L_n] \in C_{n-1}(B_n(\mathbb{Z}))$ to the integral apartment class $[S(L_1, \ldots, L_n)] \in \mathrm{St}_n\mathbb{Q}$, we get a sequence

$$C_n(BA_n(\mathbb{Z})) \longrightarrow C_{n-1}(B_n(\mathbb{Z})) \longrightarrow \mathrm{St}_n\mathbb{Q} \longrightarrow 0.$$

The Bykovskii presentation given above is equivalent to the statement that this sequence is exact. To see this, note that the relations that come from permuting the apartment are the same as those permuting the $(n-1)$-simplex in $B_n(\mathbb{Z})$, and the left most map gives exactly the Manin relation.

Proof of Church–Putman: We start with the partial projective resolution

$$C_n(BA_n(\mathbb{Z});\mathbb{Q}) \longrightarrow C_{n-1}(B_n(\mathbb{Z});\mathbb{Q}) \longrightarrow \mathbb{Q} \otimes \mathrm{St}_n\mathbb{Q} \longrightarrow 0$$

given by the Bykovskii presentation. To show that

$$H^{\binom{n}{2}-1}(\mathrm{SL}_n\mathbb{Z};\mathbb{Q}) \cong 0 \quad \text{for } n \geq 3,$$

it suffices to show that

$$C_n(BA_n(\mathbb{Z});\mathbb{Q})_{\mathrm{SL}_n\mathbb{Z}} \cong 0 \quad \text{for } n \geq 3.$$

We know that $C_n(BA_n(\mathbb{Z});\mathbb{Q})$ is freely generated by augmented frames $[L_0,\ldots,L_n]$. Assume that $L_i = \mathbb{Z}v_i$ and $v_0 = v_1 + v_2$. If $n \geq 3$, we can find a $g \in \mathrm{SL}_n\mathbb{Z}$ that swaps v_1 and v_2, sends v_n to $-v_n$ and fixes all other v_i. This matrix g will then fix $v_1 + v_2$, and act on $[L_0,\ldots,L_n]$ by flipping its orientation. It is therefore zero in coinvariants.

Proof of the Bykovskii presentation: For this we will use the result that

$$BA_n(\mathbb{Z}) \quad \text{is } (n-1)\text{-connected},$$

which is hard and technical, and was proved by Church–Putman. (Similar to Ash-Rudolph, Bykovskii did not originally use simplicial complexes.)

Let $BA'_n(\mathbb{Z})$ be the subcomplex of $BA_n(\mathbb{Z})$ of simplices $[L_0,\ldots,L_k]$ such that $L_0+\cdots+L_k$ is a proper summand of $\mathbb{Z}^n$. This implies that the $(n-2)$-skeleton of both complexes is the same (as $n-1$ lines can never span $\mathbb{Z}^n$). The only simplices in $C_*(BA_n(\mathbb{Z}), BA'_n(\mathbb{Z}))$ are frames and augmented frames. That means:

$$C_k(BA_n(\mathbb{Z}), BA'_n(\mathbb{Z})) \cong \begin{cases} C_n(BA_n(\mathbb{Z})) & \text{if } k = n \\ C_{n-1}(B_n(\mathbb{Z})) & \text{if } k = n-1 \\ 0 & \text{if } n \neq n, n-1 \end{cases}$$

Furthermore, the differential

$$C_n(BA_n(\mathbb{Z}), BA'_n(\mathbb{Z})) \longrightarrow C_{n-1}(BA_n(\mathbb{Z}), BA'_n(\mathbb{Z}))$$

is given by

$$\partial[L_0,\ldots,L_n] = [L_1,\ldots,L_n] - [L_0,L_2,\ldots,L_n] + [L_0,L_1,L_3,\ldots,L_n]$$

because

$$[L_0,L_1,L_2,\ldots,\widehat{L_i},\ldots,L_n] \in C_n(BA'_n(\mathbb{Z}))$$

for $i \geq 3$.

Thus Bykovskii gives a presentation of $H_{n-1}(BA_n(\mathbb{Z}), BA'_n(\mathbb{Z}))$. Or in other words, it suffices to show that

$$H_{n-1}(BA_n(\mathbb{Z}), BA'_n(\mathbb{Z})) \cong \mathsf{St}_n(\mathbb{Q}).$$

To do this, we will use three isomorphisms:

$$H_{n-1}(BA_n(\mathbb{Z}), BA'_n(\mathbb{Z})) \to \widetilde{H}_{n-2}(BA'_n(\mathbb{Z})) \to \widetilde{H}_{n-2}(\operatorname{sd} BA'_n(\mathbb{Z})) \to \widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q})) = \mathsf{St}_n(\mathbb{Q})$$

The first of these isomorphisms is the boundary map in the long exact sequence

$$0 = \widetilde{H}_{n-1}(BA_n(\mathbb{Z})) \to H_{n-1}(BA_n(\mathbb{Z}), BA'_n(\mathbb{Z})) \to \widetilde{H}_{n-2}(BA'_n(\mathbb{Z})) \to \widetilde{H}_{n-2}(BA_n(\mathbb{Z})) = 0.$$

The second isomorphism simply reflects that the barycentric subdivision of a simplicial complex is homeomorphic to the original complex.

The third isomorphism is more complicated. Here, every vertex in $\operatorname{sd} BA'_n(\mathbb{Z})$ is given by a simplex in $BA'_n(\mathbb{Z})$, i.e. $[L_0, \dots, L_k]$ such that $L_0 + \cdots + L_k \neq \mathbb{Z}^n$. We will send this vertex to $\mathbb{Q}(L_0 + \cdots + L_k)$, which is a vertex in $\mathcal{T}_n(\mathbb{Q})$. It is easy to check that this gives a simplicial map. It remains to check that this map induces an isomorphism on $\widetilde{H}_{n-2}$.

This can be done by a lemma of Quillen (similar to Jenny's talk) or directly by looking at the map-of-poset spectral sequence, which we will need in the next lecture anyway. For this we will use the homology of posets with coefficient functors. We will not go into the definition of this here but instead just point out that the homology of a poset with constant coefficients is the same as the homology of the simplicial complex of chains in that poset. For more details on the map-of-poset spectral sequence, please see the appendix.

Let X be the poset of simplices in $BA'_n(\mathbb{Z})$ and let Y be the poset of proper nonzero subspaces of $\mathbb{Q}^n$ ordered by inclusion. Let

$$\begin{aligned} f\colon X &\longrightarrow Y \\ [L_0, \dots, L_k] &\longmapsto \mathbb{Q}(L_0 + \cdots + L_k) \end{aligned}$$

be the span map, which is clearly a map of posets. Let V be a proper nonzero subspace of $\mathbb{Q}^n$, then the poset fiber $f_{\leq V}$ is defined as

$$f_{\leq V} = \{[L_0, \dots, L_k] \mid \mathbb{Q}(L_0 + \cdots + L_k) \subseteq V\}.$$

It is easy to see that $f_{\leq V}$ is isomorphic to the poset of simplices in $BA_{\dim V}(\mathbb{Z})$ and thus

$$\widetilde{H}_*(f_{\leq V}) \cong \widetilde{H}_*(BA_{\dim V}(\mathbb{Z})).$$

The *(reduced) map-of-poset spectral sequence* is given by

$$E^2_{pq} = H_p(Y; V \mapsto \widetilde{H}_q(f_{\leq V})) \implies H_{p+q+1}(\operatorname{cone}(f)).$$

As stated above
$$\widetilde{H}_q(f_{\leq V}) \cong \widetilde{H}_q(BA_{\mathrm{dim}V}(\mathbb{Z})).$$
In particular, this vanishes unless the dimension of V is q. For functors that are supported on an antichain, a lemma of Charney says that

$$H_p(Y; V \mapsto \widetilde{H}_q(f_{\leq V})) \cong \bigoplus_{\mathrm{dim}V=q} \widetilde{H}_{p-1}(Y_{>V}) \otimes \widetilde{H}_q(f_{\leq V}),$$

where $Y_{>V}$ is the poset of all proper subspaces of $\mathbb{Q}^n$ that are strictly including V. It is not hard to see that this is isomorphic to the poset of proper nonzero subspaces of $\mathbb{Q}^n/V \cong \mathbb{Q}^{n-\mathrm{dim}V}$. Therefore

$$\widetilde{H}_{p-1}(Y_{>V}) \cong 0 \quad \text{unless } p-1 = (n-\mathrm{dim}V)-2$$

or equivalently unless $p+q+1=n$. This means that

$$H_{p+q+1}(\mathrm{cone}(f)) \cong 0 \quad \text{unless } p+q+1=n.$$

From the long exact sequence, we get that

$$\widetilde{H}_k(X) \longrightarrow \widetilde{H}_k(Y) \quad \text{is an isomorphism for } k \leq n-2.$$

This proves the third isomorphism.

6 Patzt Lecture 3: Codimension 0 cohomology of $\Gamma_n(p)$

Goal: What is $H^{\binom{n}{2}}(\Gamma_n(p); \mathbb{Z})$?

First idea: From Borel–Serre duality we know

$$H^{\binom{n}{2}}(\Gamma_n(p); \mathbb{Z}) \cong H_0(\Gamma_n(p); \mathrm{St}_n(\mathbb{Q})) \cong \mathrm{St}_n(\mathbb{Q})_{\Gamma_n(p)}.$$

Bykovskii's presentation of Steinberg is

$$C_n(BA_n(\mathbb{Z}), BA_n'(\mathbb{Z}) \longrightarrow C_{n-1}(BA_n(\mathbb{Z}), BA_n'(\mathbb{Z})) \longrightarrow \mathrm{St}_n(\mathbb{Q}) \longrightarrow 0.$$

Because taking coinvariants is right exact, we have presentation of the coinvariants of Steinberg

$$C_n(BA_n(\mathbb{Z}), BA_n'(\mathbb{Z}))_{\Gamma_n(p)} \longrightarrow C_{n-1}(BA_n(\mathbb{Z}), BA_n'(\mathbb{Z}))_{\Gamma_n(p)} \longrightarrow \mathrm{St}_n(\mathbb{Q})_{\Gamma_n(p)} \longrightarrow 0.$$

And even more,

$$C_*(BA_n(\mathbb{Z}), BA'_n(\mathbb{Z}))_{\Gamma_n(p)} \cong C_*(BA_n(\mathbb{Z})/\Gamma_n(p), BA'_n(\mathbb{Z})/\Gamma_n(p)),$$

where $BA_n(\mathbb{Z})/\Gamma_n(p)$ and $BA'_n(\mathbb{Z})/\Gamma_n(p)$ are finite simplicial complexes.

So does that mean we are done? Maybe not. For example, what are the dimensions?

Another idea: Consider the map

$$\mathcal{T}_n(\mathbb{Q}) \longrightarrow \mathcal{T}_n(\mathbb{Q})/\Gamma_n(p).$$

This induces map

$$\mathrm{St}_n(\mathbb{Q}) = \widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q})) \longrightarrow \widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q})/\Gamma_n(p))$$

that factors through

$$\mathrm{St}_n(\mathbb{Q})_{\Gamma_n(p)} \longrightarrow \widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q})/\Gamma_n(p))$$

because the above map is $\Gamma_n(p)$-equivariant with the trivial action of $\widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q})/\Gamma_n(p))$. (This is also finite simplicial complex homotopy equivalent to a wedge of $(n-2)$-spheres.)

What can we say about this map?

Miller–P–Putman: For $p \geq 3$ a prime,

$$H^{\binom{n}{2}}(\Gamma_n(p);\mathbb{Z}) \longrightarrow \widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q})/\Gamma_n(p))$$

is always surjective.
For $p \leq 5$, it is injective.
For $p \geq 7$ (and $n \geq 2$), it is *not* injective.

Computations: $\widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q})/\Gamma_n(p))$ is a free abelian group and its dimension is very computational. Below is a list of all dimensions for $p = 5$ and $n \leq 15$. The numbers in the table were computed in one second. I have computed the dimensions for $p = 5$ and $n \leq 200$ in less than one minute on a personal computer.

n	$\dim H^{\binom{n}{2}}(\Gamma_n(5))$
1	1
2	11
3	621
4	176331
5	250654141
6	1781972405051
7	63346001119010061
8	1125931261576107996017 1
9	10006344346503001479394156381
10	4446406792276999676003075050900969 1
11	9878999911070267785826675889958592705411 01
12	1097455152004635612974384057874082942100009044816 11
13	6095798286516944110137857138523470278325534103710325837222 1
14	169295103797089744818524470008237065225058191012577153712309414663931
15	2350867829470159774034814041007591566603522538519291648712545382850352884817741

Case $n=2$**:** $B\Gamma_2(p)$ is the modular curve of this congruence subgroup. That means it is a orientable connected surface. For $p \geq 3$, it is known that it has genus

$$\frac{(p+2)(p-3)(p-5)}{24}$$

and it is easy to see that its punctures are in bijection to $\mathcal{T}_2(\mathbb{Q})/\Gamma_2(p)$. In particular, that means

$$H^1(\Gamma_2(p);\mathbb{Z}) \cong \mathbb{Z}^{\frac{(p+2)(p-3)(p-5)}{12}} \oplus \widetilde{H}_0(\mathcal{T}_2(\mathbb{Q})/\Gamma_2(p)).$$

The theorem is evident from this.

Connectivity result: We prove that $BA_n(\mathbb{Z})/\Gamma_n(p)$ is

- $(n-2)$-connected for all primes p, and even
- $(n-1)$-connected for primes $p \leq 5$.

Proof of surjectivity: Because

$$C_i(BA_n(\mathbb{Z})/\Gamma_n(p), BA'_n(\mathbb{Z})/\Gamma_n(p)) \cong 0 \quad \text{for } i \leq n-2,$$

the initial observations show that

$$H_{n-1}(BA_n(\mathbb{Z})/\Gamma_n(p), BA'_n(\mathbb{Z})/\Gamma_n(p)) \cong \mathrm{St}_n(\mathbb{Q})_{\Gamma_n(p)}.$$

Using the long exact sequence of a pair, we get that the boundary map

$$H_{n-1}(BA_n(\mathbb{Z})/\Gamma_n(p), BA'_n(\mathbb{Z})/\Gamma_n(p)) \longrightarrow \widetilde{H}_{n-2}(BA'_n(\mathbb{Z})/\Gamma_n(p))$$

is surjective because

$$\widetilde{H}_{n-2}(BA_n(\mathbb{Z})/\Gamma_n(p)) \cong 0.$$

Now again, we can replace the homology of $BA'_n(\mathbb{Z})/\Gamma_n(p)$ with the isomorphic homology of the poset of simplices. In the last step, we consider the span map

$$\widetilde{H}_{n-2}(\operatorname{sd} BA'_n(\mathbb{Z})/\Gamma_n(p)) \longrightarrow \widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Z})/\Gamma_n(p)).$$

One can prove that this is surjective with a map-of-poset spectral sequence argument, which we will discuss later.

Proof of injectivity for $p \le 5$: We consider the same maps as in the surjectivity proof:

$$\begin{aligned} H_{n-1}(BA_n(\mathbb{Z})/\Gamma_n(p), BA'_n(\mathbb{Z})/\Gamma_n(p)) &\longrightarrow \widetilde{H}_{n-2}(BA'_n(\mathbb{Z})/\Gamma_n(p)) \\ &\cong \widetilde{H}_{n-2}(\operatorname{sd} BA'_n(\mathbb{Z})/\Gamma_n(p)) \longrightarrow \widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Z})/\Gamma_n(p)). \end{aligned}$$

For the first map, the long exact sequence of a pair shows it is an isomorphism as

$$\widetilde{H}_{n-2}(BA_n(\mathbb{Z})/\Gamma_n(p)) \cong \widetilde{H}_{n-1}(BA_n(\mathbb{Z})/\Gamma_n(p)) \cong 0 \quad \text{for primes } p \le 5.$$

Then last map can now be proved to be an isomorphism with a map-of-poset spectral sequence argument, which we will also discuss later.

Proof of non-injectivity for $p \ge 7$: To prove this part of the theorem, we will show that the span map

$$\widetilde{H}_{n-2}(\operatorname{sd} BA'_n(\mathbb{Z})/\Gamma_n(p)) \longrightarrow \widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Z})/\Gamma_n(p))$$

is not injective. This will again be done by a map-of-poset spectral sequence argument.

Description of $BA_n(\mathbb{Z})/\Gamma_n(p)$ and $\mathcal{T}_n(\mathbb{Z})/\Gamma_n(p)$: Before we get to the map-of-poset spectral sequence argument, we will have to understand the simplicial complexes $BA_n(\mathbb{Z})/\Gamma_n(p)$ and $\mathcal{T}_n(\mathbb{Z})/\Gamma_n(p)$ a little better. Both of these are almost versions of BA_n and $\mathcal{T}_n$ for $\mathbb{F}_p$, except that all matrices in $\Gamma_n(p)$ have determinant 1 and $\mathbb{F}_p$ (usually) has more units than $\mathbb{Z}$.

To be precise, $BA_n(\mathbb{Z})/\Gamma_n(p)$ is (isomorphic to) a simplicial complex whose vertices are nonzero vectors in $\mathbb{F}_p^n$ up to multiplying by -1. We write such an equivalence class of vectors as $\pm v$ for some vector $v \in \mathbb{F}_p^n$. The simplices are the subsets of set $\{\pm v_0, \dots, \pm v_n\}$ such that the matrix built from $v_1, \dots, v_n$ has determinant ± 1 and $v_0 = v_1 + v_2$ (for some choices).

$\mathcal{T}_n(\mathbb{Q})/\Gamma_n(p)$ is the simplicial complex of chains in the following poset. This poset is given by elements (V,ω), where V is a nonzero, proper subspace of $\mathbb{F}_p^n$, and

$$\omega \in \left(\bigwedge^{\dim V} V\right)/\pm 1 \cong F^\times/\pm 1.$$

The ordering in this poset is just given by inclusion, independent of the ω.

The map-of-poset spectral sequence argument:

Let X be the poset of simplices in $BA_n'(\mathbb{Z})/\Gamma_n(p)$ and let Y be the poset associated to $\mathcal{T}_n(\mathbb{Q})/\Gamma_n(p)$ from above. Let

$$\begin{aligned} f\colon X &\longrightarrow Y \\ [v_0,\ldots,v_k] &\longmapsto (V,\omega) = (\mathbb{F}_p\cdot v_0 + \cdots + \mathbb{F}_p \cdot v_k, v_0\wedge\cdots\wedge v_k) \end{aligned}$$

be the span map, which is clearly a map of posets. Let V be a proper nonzero subspace of $\mathbb{F}_p^n$ and $\omega \in \left(\bigwedge^{\dim V} V\right)/\pm 1$, then the poset fiber $f_{\le (V,\omega)}$ is defined as

$$f_{\le V} = \{[v_0,\ldots,v_k] \mid \mathbb{F}_p\cdot v_0 + \cdots + \mathbb{F}_p\cdot v_k \subseteq V \text{ and } v_0\wedge\cdots\wedge v_k = \omega \text{ if the span is all of } V\}.$$

Now, one can see that $f_{\le V}$ is isomorphic to the poset of simplices in $BA_{\dim V}(\mathbb{Z})/\Gamma_{\dim V}(p)$ and thus

$$\widetilde{H}_*(f_{\le V}) \cong \widetilde{H}_*(BA_{\dim V}(\mathbb{Z})/\Gamma_{\dim V}(p)).$$

The *(reduced) map-of-poset spectral sequence* is given by

$$E^2_{pq} = H_p(Y; V \mapsto \widetilde{H}_q(f_{\le V})) \implies H_{p+q+1}(f).$$

Differently than in the previous lecture,

$$\widetilde{H}_q(f_{\le V}) \cong \widetilde{H}_q(BA_{\dim V}(\mathbb{Z})/\Gamma_{\dim V}(p))$$

might be nonzero when the dimension of V is q or $q+1$. Therefore we will have to split this functor up into two:

$$0 \longrightarrow A_q(V) \longrightarrow \widetilde{H}_q(f_{\le V}) \longrightarrow B_q(V) \longrightarrow 0,$$

where

$$A_q(V) = \begin{cases} \widetilde{H}_q(f_{\le V}) & \text{if } \dim V = q, \\ 0 & \text{otherwise} \end{cases}$$

and

$$B_q(V) = \begin{cases} \widetilde{H}_q(f_{\le V}) & \text{if } \dim V = q+1, \\ 0 & \text{otherwise.} \end{cases}$$

Using the lemma of Charney, we get

$$H_p(Y; A_q) \cong \bigoplus_{\dim V = q} \widetilde{H}_{p-1}(Y_{>V}; \widetilde{H}_q(f_{\leq V}))$$

and

$$H_p(Y; B_q) \cong \bigoplus_{\dim V = q+1} \widetilde{H}_{p-1}(Y_{>V}; \widetilde{H}_q(f_{\leq V})).$$

Here $Y_{>V}$ is isomorpic to $\mathcal{T}_{n-\dim V}(\mathbb{Q})/\Gamma_{n-\dim V}(p)$. Therefore

$$\widetilde{H}_{p-1}(Y_{>V}) \cong 0 \quad \text{unless } p-1 = (n - \dim V) - 2.$$

This means that

$$H_p(Y; A_q) \cong 0 \quad \text{unless } p+q = n-1$$

and

$$H_p(Y; B_q) \cong 0 \quad \text{unless } p+q = n-2.$$

Using the long exact sequence we now get

$$E^2_{pq} = \begin{cases} H_p(Y; A_q) & \text{for } p+q = n-1, \\ H_p(Y; B_q) & \text{for } p+q = n-2, \\ 0 & \text{otherwise.} \end{cases}$$

We can deduce that

$$H_{p+q+1}(f) \cong 0 \quad \text{unless } p+q+1 \in \{n, n-1\}.$$

From the long exact sequence, we get that

$$\widetilde{H}_k(X) \longrightarrow \widetilde{H}_k(Y) \quad \text{is an isomorphism for } k \leq n-3$$

and

$$\widetilde{H}_{n-2}(X) \longrightarrow \widetilde{H}_{n-2}(Y) \quad \text{is surjective.}$$

This proves the surjection part of the theorem.

Now, assuming $p \leq 5$, $B_q = 0$, so we get

$$\widetilde{H}_k(X) \longrightarrow \widetilde{H}_k(Y) \quad \text{is an isomorphism for } k \leq n-2.$$

This proves the injection part of the theorem for $p \leq 5$.

Assuming $p \geq 7$ and get non-injectivity, we have to take a closer look at the spectral sequence. We want to prove that $E^\infty_{n-3,1} \neq 0$. This would follow from $E^2_{n-3,1} \neq 0$ and $E^2_{n-1,0} = 0$. We already saw that

$$E^2_{n-3,1} \cong H_{n-3}(Y; B_1) \cong \bigoplus_{\dim V=2} \widetilde{H}_{n-4}(\mathcal{T}_{n-2}(\mathbb{Q})/\Gamma_{n-2}(p)) \otimes \widetilde{H}_1(BA_2(\mathbb{Z})/\Gamma_2(p)),$$

which is nonzero for $p \geq 7$. And we can also find that

$$E^2_{n-1,0} \cong H_{n-1}(Y; A_0) \cong 0$$

because Y has only dimension $n-2$ (or alternatively $A_0 \cong \widetilde{H}_0(\emptyset)) \cong 0$).

This implies that $H_{n-1}(f) \neq 0$ and the long exact sequence

$$0 = \widetilde{H}_{n-1}(Y) \longrightarrow H_{n-1}(f) \longrightarrow \widetilde{H}_{n-2}(X) \longrightarrow \widetilde{H}_{n-2}(Y) \longrightarrow \dots$$

shows that the map

$$\widetilde{H}_{n-2}(\mathrm{sd}\, BA'_n(\mathbb{Z})/\Gamma_n(p)) \cong \widetilde{H}_{n-2}(X) \longrightarrow \widetilde{H}_{n-2}(Y) \cong \widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Z})/\Gamma_n(p))$$

is not injective for $p \geq 7$.

7 Further directions and open problems

The topic of high dimensional cohomology arithmetic groups has 4 obvious directions in which one can go. The following cases are already known.

Codimension: We covered the Church–Farb–Putman conjecture

$$H^{\binom{n}{2}-i}(\mathrm{SL}_n\mathbb{Z}; \mathbb{Q}) \cong 0 \quad \text{for } i \leq n-2$$

in the codimensions $i \leq 1$ as theorems of Lee–Szczarba and Church–Putman. For $i = 2$, the conjecture has recently also been proven correct by Brück–Miller–P–Sroka–W. The cases $i \geq 3$ are open and will either require more significantly more computer calculations or new ideas.

Rings: In codimension $i = 0$, we actually know

$$H^{vcd_n}(\mathrm{SL}_n\mathcal{O}_K; \mathbb{Q}) \cong 0 \quad \text{for } n \geq 2$$

for all number rings $\mathcal{O}_K$ that are Euclidean by Lee–Szczarba. For number rings that are not PIDs, Church–Farb–Putman showed that

$$\dim H^{vcd_n}(\mathrm{SL}_n\mathcal{O}_K; \mathbb{Q}) \geq (|cl(\mathcal{O}_K)| - 1)^{n-1}.$$

Assuming the Generalized Riemann Hypothesis, there are only four number rings $\mathcal{O}_K$ that are PIDs but not Euclidean: $K = \mathbb{Q}[\sqrt{-19}], \mathbb{Q}[\sqrt{-43}], \mathbb{Q}[\sqrt{-67}], \mathbb{Q}[\sqrt{-163}]$. Miller–P–W–Yasaki proved that

$$H^{vcd_{2n}}(\mathrm{SL}_{2n}\mathcal{O}_K;\mathbb{Q}) \neq 0 \quad \text{for } 2n \geq 2$$

for the three latter rings. The odd n cases and $K = \mathbb{Q}[\sqrt{-19}]$ are still open. ($H^{vcd_2}(\mathrm{SL}_2\mathcal{O}_K;\mathbb{Q}) \cong 0$ for $K = \mathbb{Q}[\sqrt{-19}]$.)

Combing the directions of codimension and rings very little is known: Kupers–Miller–P–W proved in codimension 1 that

$$H^{vcd_n - 1}(\mathrm{SL}_n\mathcal{O}_K;\mathbb{Q}) = 0 \quad \text{for } n \geq 3$$

for $\mathcal{O}_K$ being the Gaussian and the Eisenstein integers.

Subgroups: Only the prime level congruence subgroups of $\mathrm{SL}_n\mathbb{Z}$ have been considered so far. Lee–Szczarba proved that

$$H^{vcd_n}(\Gamma_n(3);\mathbb{Z}) \cong \mathrm{St}_n\mathbb{F}_3.$$

Miller–P–Putman extended a generalization of this result

$$H^{vcd_n}(\Gamma_n(p);\mathbb{Z}) \cong \widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q})/\Gamma_n(p);\mathbb{Z})$$

to the primes $p \leq 5$ and disproved it for primes $p \geq 7$. In Miller–P–Putman, a lower bound for $p \geq 7$ is given. This lower bound since has been improved by Schwermer.

Generally, it is not hard to see that

$$\mathrm{Ind}_{\mathrm{GL}_n(\mathbb{F}_p)}^{\mathrm{GL}_{n+1}(\mathbb{F}_p)} H^{vcd_n}(\Gamma_n(p);\mathbb{Z}) \longrightarrow H^{vcd_{n+1}}(\Gamma_{n+1}(p);\mathbb{Z})$$

is a surjection. Miller–Nagpal–P consider this also statement also in codimension $i = 1$ and prove it for $p = 3$:

$$\mathrm{Ind}_{\mathrm{GL}_n(\mathbb{F}_3)}^{\mathrm{GL}_{n+1}(\mathbb{F}_3)} H^{vcd_n - 1}(\Gamma_n(3);\mathbb{Z}) \longrightarrow H^{vcd_{n+1} - 1}(\Gamma_{n+1}(3);\mathbb{Z})$$

is a surjection. This uses that

$$H^{vcd_n}(\Gamma_n(3);\mathbb{Z}) \cong \mathrm{St}_n\mathbb{F}_3$$

in a critical way.

Arithmetic groups: Other than SL_n, one can of course also consider other arithmetic groups. So far only the symplectic groups $\mathrm{Sp}_{2n}\mathbb{Z}$ have been considered. Building on a result of Gunnells, it is easy to observe that

$$H^{vcd_n}(\mathrm{Sp}_{2n}\mathcal{O}_K;\mathbb{Q}) \cong 0 \quad \text{for } n \geq 1$$

for all Euclidean number rings $\mathcal{O}_K$.

Brück–P–Sroka have work-in-progress on the codimension $i = 1$ case

$$H^{vcd_n - 1}(\mathrm{Sp}_{2n}\mathbb{Z}; \mathbb{Q}) \cong 0 \quad \text{for } n \geq 2.$$

Open problems

All other combinations of directions are still open. Although a statement about all congruence subgroups of all arithmetic groups over all number rings in all codimensions would be desirable, that seems in no way feasible or unifiable. The most interesting open problems that seem at least somewhat achievable (albeit with new ideas) are:

- The Church–Farb–Putman conjecture

  $$H^{\binom{n}{2} - i}(\mathrm{SL}_n\mathbb{Z}; \mathbb{Q}) \cong 0 \quad \text{for } i \leq n - 2$$

  in all codimensions i. (Potentially even for all Euclidean number rings instead of only $\mathbb{Z}$.)

- The symplectic version of the Church–Farb–Putman conjecture

  $$H^{vcd_n - i}(\mathrm{Sp}_{2n}\mathbb{Z}; \mathbb{Q}) \cong 0 \quad \text{for } i \leq n - 1$$

  in all codimensions i. (Potentially even for all Euclidean number rings instead of only $\mathbb{Z}$.)

- Calculating

  $$H^{vcd_n}(\mathrm{SL}_n\mathcal{O}_K; \mathbb{Q})$$

  when $\mathcal{O}_K$ is not Euclidean. (Here we don't even have a conjecture. Except for maybe the four number rings that are PIDs but not Euclidean. More about that in a little bit.)

- Calculating

  $$H^{vcd_n}(\Gamma_n(p); \mathbb{Q})$$

  for all primes $p \geq 7$.

More tractable ideas would be the following:

- Let $\mathcal{O}_K$ be a Euclidean number ring. Define $BP_n(\mathcal{O}_K)$ to be the simplicial complex whose vertices are 1- and 2-dimensional summands of $\mathcal{O}_K^n$ and top-dimensional simplices are given by $\{P, L_1, \ldots, L_n\}$, where

  $$\mathcal{O}_K^n = L_1 \oplus \cdots \oplus L_n \quad \text{and} \quad P = L_1 \oplus L_2.$$

  If one could prove that $BP_n(\mathcal{O}_K)$ is $(n-1)$-connected for $n \geq 2$, it would show that

  $$H^{vcd_n - 1}(\mathrm{SL}_n\mathcal{O}_K; \mathbb{Q}) \cong 0 \quad \text{for } n \geq 3.$$

- Let $\mathcal{O}_K$ be a one of the four number rings mentioned above that are PIDs but not number rings. If one could prove that $BP_n(\mathcal{O}_K)$ is $(n-2)$-connected for $n \geq 3$, one would be able to calculate
$$H^{vcd_n}(\mathrm{SL}_n \mathcal{O}_K; \mathbb{Q}).$$
- What other congruence subgroups $\Gamma_2 \leq \mathrm{SL}_2\mathbb{Z}$ have a modular curve with genus 0? Is there a way to naturally build a sequence
$$\Gamma_2 \leq \Gamma_3 \leq \Gamma_4 \leq \dots$$
with $\Gamma_n \leq \mathrm{SL}_n\mathbb{Z}$ such that
$$H^{\binom{n}{2}}(\Gamma_n; \mathbb{Q}) \cong \widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q})/\Gamma_n; \mathbb{Q})$$
or one at least gets a surjection?
- Using the Bykovskii presentation of Steinberg for the Gaussian integers and the Eisenstein integers enables us to give a presentation of the top cohomology of the congruence subgroups Γ of $\mathrm{SL}_n\mathbb{Z}[i]$ and $\mathrm{SL}_n\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. One might want to compare these to $\widetilde{H}_{n-2}(\mathcal{T}(K)/\Gamma)$ where K is either $\mathbb{Q}[i]$ or $\mathbb{Q}[\sqrt{-3}]$.
- Can one utilize the presentation of Brück–P–Sroka of the symplectic Steinberg module to get a Miller–P–Putman-type result for principle congruence subgroups of $\mathrm{Sp}_{2n}\mathbb{Z}$? (The simplicial complexes involved for this question seem very technical but maybe $p = 3$ is actually not so hard.)
- Knowing $H^{\binom{n}{2}}(\Gamma_n(5))$ (by Miller–P–Putman), it seems tractable to ask if
$$\mathrm{Ind}_{\mathrm{GL}_n\mathbb{F}_5}^{\mathrm{GL}_{n+1}\mathbb{F}_5} H^{\binom{n}{2}-1}(\Gamma_n(5)) \longrightarrow H^{\binom{n+1}{2}-1}(\Gamma_{n+1}(5))$$
is a surjection.
- As far as I know, the Hecke action/Hecke eigenvalues for $H^{vcd}(\Gamma_n(5))$ have not been calculated yet.

8 Exercises

Contents

8.1 Symmetric spaces and horoballs

Exercise 24. Let $A, B \in \mathrm{SL}_n\mathbb{R}$. Check that AA^T is a positive definite symmetric matrix. Check that every positive definite symmetric matrix has this form. Check that $AA^T = BB^T$ if and only if $B^{-1}A \in \mathrm{SO}(n)$.

Exercise 25. What is the action of $\mathrm{SL}_2\mathbb{Z}$ on $\mathbb{H}_2$ induced by the homeomorphism $X_2 \cong \mathbb{H}_2$?

Exercise 26. For each rational number $q = \frac{a}{b} \in \mathbb{Q}$ with $a, b \in \mathbb{Z}$ that are coprime, let there be a circle tangent to the real number line in $\mathbb{C}$ at $\frac{a}{b}$ from above with diameter $\frac{1}{b^2}$. (Call this the circle "at $\frac{a}{b}$"). These are call *Ford circles*. They are examples of hyperbolic *horocycles*.

(a) Prove that the circle at $\frac{1}{2}$ is exactly the circle tangent to the circle at 0, the circle at 1, and the real line.

(b) Prove that the circle at $\frac{1}{3}$ is exactly the circle that touches the circle at 0, the circle at $\frac{1}{2}$, and the real line.

(c) Show in general that none of these circles intersect transversely and that the two circles at $\frac{a}{b}$ and $\frac{c}{d}$ touch if and only if

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1.$$

Exercise 27. We consider the Ford circles under the action of $\mathrm{SL}_2\mathbb{Z}$.

(a) Show that the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\mathbb{Z}$$

sends the circle at 0 to the circle at $\frac{b}{d}$ (and the horizontal line through i if $d = 0$).

(b) If one shrinks the circle at 0 by the factor $\lambda > 0$, by what factor is its $\mathrm{SL}_2\mathbb{Z}$-image at $\frac{b}{d}$ shrunk?

8.2 The Tits buildings

Let $\mathbb{F}$ be a field, and V an $\mathbb{F}$-vector space. Recall that the *Tits building* $\mathcal{T}(V)$ is the geometric realization of the poset of proper nonzero vector subspaces of V under inclusion. We sometimes write $\mathcal{T}_n(\mathbb{F})$ for $\mathcal{T}(\mathbb{F}^n)$.

With the following exercises, we can extend the definition of the Tits building to PIDs.

Exercise 28. Let R be a PID, and $U \subseteq R^n$ be an R-submodule. Show that the following conditions are equivalent. If these conditions hold, we say that U is *split* or that U is a *summand* of R^n.

(i) There exists an R-submodule C such that $R^n = U \oplus C$.

(ii) There exists a basis for U that extends to a basis for R^n.

(iii) Any basis for U extends to a basis for R^n.

(iv) The quotient R^n/U is a free R-module.

Exercise 29. Let R be a PID. An element $v = (r_1, r_2, \ldots, r_n) \in R^n$ is called *unimodular* if the ideal generated by $(r_1, r_2, \ldots, r_n)$ is R. In other words, the gcd of $(r_1, r_2, \ldots, r_n)$ is a unit.

(a) Show that a nonzero element $v \in R^n$ spans a direct summand of R^n if and only if it is unimodular.

(b) Show that $v \in R^n$ is an element of a basis for R^n if and only if it is unimodular.

(c) Give an example of a PID R and two unimodular vectors in R^n that can never both be elements of the same basis.

For R a PID and V a free R-module, we may further define $\mathcal{T}(V)$ to be the poset of proper nonzero summands of V under inclusion.

Exercise 30. Let R be a PID, and let U, V be summands of R^n.

(a) Show that $U \cap V$ is always a summand of R^n.

(b) Show that $U + V$ need not be a summand of R^n.

Exercise 31. Let $U \subseteq R^n$ be a direct summand, and let $W \subseteq U$. Show that W is a summand of U if and only if it is a summand of R^n.

In the next exercises, we will show that this ostensible generalizations of the Tits buildings to PIDs in fact reduces to the field case.

Exercise 32. Let R be a PID and $F(R)$ be its field of fractions.

(a) Show that there is a bijection between R-submodule summands of R^n and $F(R)$-vector subspaces of $F(R)^n$, given by the following correspondence. Consider R^n as a subset of $F(R)^n$.

$$\begin{aligned} \{\text{summands of } R^n\} &\longleftrightarrow \{\text{subspaces of } F(R)^n\} \\ U &\longmapsto F(R)\text{-span of } U \\ V \cap R^n &\longleftarrow\!\shortmid V \end{aligned}$$

(b) Verify that this bijection induces an isomorphism of posets of submodules under inclusion.

(c) Conclude that $\mathcal{T}_n(R)$ can be canonically identified with $\mathcal{T}_n(F(R))$.

8.3 Coxeter complexes and buildings

The Tits buildings, as the name suggests, are examples of *buildings*. In this exercise we will define a building and verify that the Tits buildings satisfy the axioms.

To define a building, we first need the notion of a Coxeter complex.

Definition (Coxeter system). A *Coxeter system* is a group W (the *Coxeter group*) along with a distinguished generating set $S = \{s_1, s_2, \ldots, s_n\} \subseteq W$ such that the corresponding presentation has the form

$$W = \langle s_1, s_2, \ldots, s_n \mid (s_i s_j)^{m_{i,j}} = 1 \rangle, \qquad m_{i,i} = 1, \quad 2 \leq m_{i,j} \leq \infty \text{ for } i \neq j.$$

Coxeter groups are abstract generalizations of *reflection groups*. Notably for our purposes, the symmetric group S_{n+1} is a Coxeter group with generators the simple transpositions $s_i = (i\ i+1)$ and associated presentation

$$S_{n+1} = \langle s_1, s_2, \ldots, s_n \mid s_i^2,\ (s_i s_{i+1})^3,\ (s_i s_j)^2 \text{ for } |i-j| > 1 \rangle.$$

Given a Coxeter system (W, S), a *standard subgroup* of W is any subgroup W_J generated by a subset J of S. A *standard coset* is a coset wW_J for $w \in W$ and W_J a standard coset.

Definition (Coxeter complex). Given a Coxeter system (W, S), consider the poset $P_{W,S}$ of proper standard cosets under reverse inclusion. One way to define the *Coxeter complex* $X_{W,S}$ associated to (W, S) is as follows. The p-simplices of $X_{W,S}$ are indexed by standard cosets wW_J with $|J| = n-p-1$, and assembled in such a way that the geometric realization of $P_{W,S}$ is the barycentric subdivision of $X_{W,S}$. In other words, the poset of cells of $X_{W,S}$ under inclusion is precisely $P_{W,S}$.

Exercise 33. (a) Sketch the Coxeter complex for the symmetric groups S_2 and S_3.

(b) Describe the standard cosets in the symmetric group S_{n+1}.

(c) Show that the Coxeter complex of S_{n+1} can be identified with the flag complex of nonempty subsets of the set $\{1, 2, \ldots, n, n+1\}$.

(d) Show that the Coxeter complex of the symmetric group S_{n+1} can be identified with the boundary of a barycentrically-subdivided n-simplex. Conclude that the Coxeter complex is topologically a sphere S^{n-1}.

Definition (Building). A *building* is a simplicial complex Δ that can be written as a union of subcomplexes Σ, called *apartments*, that satisfy the following axioms.
(B0) Each apartment Σ is a Coxeter complex.
(B1) For any two simplices $A, B \in \Delta$, there is an apartment Σ containing both of them.
(B2) If Σ and Σ' are two apartments containing A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.

Top-dimensional simplices are called *chambers*, and codimension-one simplices are *panels*.

Condition **(B2)** is equivalent to the following.

(B2′) Let Σ and Σ' be two apartments containing a simplex C that is a chamber of Σ. Then there is an isomorphism $\Sigma \xrightarrow{\cong} \Sigma'$ fixing every simplex of $\Sigma \cap \Sigma'$.

Let V be a vector space over $\mathbb{Q}$. Recall that the *Tits buildings* $\mathcal{T}(V)$ is the geometric realization of the poset of proper nonzero vector subspaces of V.

Further recall that a *frame* for V is a decomposition $V = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ of V as a direct sum of 1-dimensional subspaces L_i. For each frame $L = \{L_1, L_2, \cdots, L_n\}$ for V, we define an apartment A_L to be the full subcomplex of $\mathcal{T}(V)$ on vertices corresponding to direct sums of all proper nonempty subsets of $\{L_1, L_2, \cdots, L_n\}$.

In the next exercise we will verify that the Tits building $\mathcal{T}(V)$, along with the system of apartments $\{A_L \mid L \text{ a frame for } V\}$, is a building.

Exercise 34. (a) Suppose V is n-dimensional. Show that an apartment A_L is isomorphic to the Coxeter complex associated to S_n. Conclude that axiom **(B0)** holds.

(b) Given a flag $0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_p \subsetneq V$, let's say that frame $L = \{L_1, L_2, \cdots, L_n\}$ is *compatible* with this flag if every subspace V_i is a direct sum of lines L_j. Show that, given any two flags in V, there is a frame that is compatible with both of them. Use this result to conclude that **(B1)** holds.
(Despite being "just" elementary linear algebra, this exercise is not entirely trivial! See Abramenko–Brown [AB, Section 4.3].)

(c) Verify that axiom **(B2')** holds.
Hint: Given a chamber in Σ corresponding to a complete flag $0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V$, construct an explicit isomorphism to the Coxeter complex using the function

$$\begin{aligned} \phi : \Sigma &\longrightarrow \{\text{subsets of } [n]\} \\ U &\longmapsto \{i \mid \dim(U \cap V_i) < \dim(U \cap V_{i+1})\} \end{aligned}$$

Observe that this isomorphism depends only on the chamber and not on Σ.

8.4 The Solomon–Tits Theorem

The goal of this section is to give a proof of the Solomon–Tits theorem, which states that the Tits building $\mathcal{T}_n(K)$ is homotopy equivalent to a wedge of spheres of dimension $(n-2)$.

Fix a field K and a positive integer n. Recall that the Tits building $\mathcal{T}_n(K)$ is the geometric realization of the poset of proper nonzero subspaces of K^n under inclusion. Explicitly, $\mathcal{T}_n(K)$ is a simplicial complex defined as follows. The vertices of $\mathcal{T}_n(K)$ are proper nonzero subspaces of K^n. A collection of vertices span a simplex precisely when they form a flag.

Exercise 35. Fix a field K. Verify that, when $n = 1$, the building $\mathcal{T}_n(K)$ is empty, and when $n = 2$, the building $\mathcal{T}_n(K)$ is a discrete set of points, that is, a wedge of 0-spheres.

Exercise 36.

Draw the Tits building for $K = \mathbb{Z}/2\mathbb{Z}$ and $n \leq 3$. Can you see explicitly that it is homotopy equivalent to a wedge of spheres?

To pove the Solomon–Tits theorem, will use a method sometimes called "discrete Morse theory". I first learned this proof from Bestvina's notes [Be].

Definition (Realizations and links). For a poset T, write $|T|$ for its geometric realization. For $t \in T$, we write $\mathrm{Lk}_T(t)$ for the link of t in T,

$$\mathrm{Lk}_T(t) = \{s \in T \mid s < t \text{ or } s > t\}.$$

We write $\mathrm{Lk}_T^{\uparrow}(t)$ for the subposet

$$\mathrm{Lk}_T^{\uparrow}(t) = \{s \in T \mid s > t\}$$

and we write $\mathrm{Lk}_T^{\downarrow}(t)$ for the subposet

$$\mathrm{Lk}_T^{\downarrow}(t) = \{s \in T \mid s < t\}.$$

Exercise 37.
Verify that $|\mathrm{Lk}_T(t)| = |\mathrm{Lk}_T^{\uparrow}(t)| \underset{\text{join}}{*} |\mathrm{Lk}_T^{\downarrow}(t)|$.

The following result is our key lemma.

Lemma (Discrete Morse Theory). Let T be a poset with $T = X_0 \cup T_1 \cup \cdots \cup T_m$ as sets. Let $X_k = X_0 \cup T_1 \cup \cdots \cup T_k$. Suppose the following:
(i) $|X_0|$ is contractible.
(ii) For $i \geq 1$ then any pair $s, t \in T_i$ of distinct elements are not comparable.
(iii) For $i \geq 1$ and $t \in T_i$,

$$|\mathrm{Lk}_T(t) \cap X_{i-1}| \simeq \bigvee S^{d-1} \qquad \text{or} \qquad |\mathrm{Lk}_T(t) \cap X_{i-1}| \simeq *.$$

Then $|T|$ is $(d-1)$-connected. In particular, if $|\mathrm{Lk}_T(t) \cap X_{i-1}| \simeq \bigvee S^{d-1}$ for at least one i and t, then $|T|$ is homotopy equivalent to a wedge of d-spheres. Otherwise, $|T|$ is contractible.

Exercise 38. Prove the Discrete Morse Theory lemma.

Exercise 39. Fix a field K and $n \geq 2$. Let T be the poset of nonzero proper subspaces of K^n, so $\mathcal{T}_n(K)$ is defined to be $|T|$. Assume by induction that $\mathcal{T}_m(K) \simeq \bigvee S^{m-2}$ for all $m < n$; we proved the base case in Exercise (35). Fix a line L in K^n.

- Let X_0 be the subposet of T on vertices V such that $L \subseteq V$.
- For $i = 1, \ldots, n-1$, let T_i be the set of subspaces V of K^n

$$T_i = \{V \subseteq K^n \mid \dim(V) = i,\ L \not\subseteq V\ \}$$

(a) Verify that $|X_0| \subseteq |T|$ is the star on the vertex L, and hence contractible.

(b) Verify that for fixed i, distinct elements in T_i are not comparable.

(c) Suppose $1 \leq i \leq (n-2)$. Show that, for $V \in T_i$, the subspace $(V + L)$ is a cone point of $|\mathrm{Lk}_T(V) \cap X_{i-1}|$. (Why did we need the assumption $i \leq (n-2)$?)

(d) Verify that, for $i = n-1$ and $V \in T_i$,

$$|\mathrm{Lk}_T(V) \cap X_{i-1}| \simeq \bigvee S^{n-3}.$$

Hint: Compare $|\mathrm{Lk}_T(V) \cap X_{i-1}|$ to $\mathcal{T}_{n-1}(K)$.

(e) Use the Discrete Morse Theory lemma to conclude that $\mathcal{T}_n(K) \simeq \bigvee S^{n-2}$.

There are other elegant approaches to computing the homotopy type of $\mathcal{T}_n(K)$. For example, see Abramenko–Brown [AB, Section 4.12] for an approach using the theory of shellability. In principle, these proofs can also be used to describe a generating sets for the reduced homology of $\mathcal{T}_n(K)$.

8.5 The Sharbly resolution

Let R be a PID, and let $\mathrm{St}_n(R)$ be the associated Steinberg module,

$$\mathrm{St}_n(R) := \widetilde{H}_{n-2}(\mathcal{T}_n(R); \mathbb{Z}).$$

By Exercise 32, we can identify $\mathcal{T}_n(R)$ with $\mathcal{T}_n(\mathbb{F})$, where $\mathbb{F}$ is the field of fractions of R.

In this section we will construct a resolution of the Steinberg module due to Lee–Szczarba [LS], which has since been named the *Sharbly resolution*. To quote Lee–Szczarba [LS, Section 4]:

> *In theory, one should be able to use [the Sharbly resolution] to compute the groups $H_q(\mathrm{SL}_n(R); \mathrm{St}_n(R))$ for $q \geq 0$. However, because of the size of [the terms of the resolution], this is impractical except when $q = 0$.*

The proof of the Sharbly resolution is both a significant historical development, and also an instructive application of some of the techniques of the field. The construction will use the *Acyclic Covering Lemma* (See Brown "Cohomology of Groups" Section VII Lemma 4.4). Let X be a CW complex, and suppose X is the union of a family of nonempty subcomplexes

$$X = \bigcup_{\alpha \in J} X_\alpha.$$

The *nerve* N of the family X_α is the abstract simplicial complex with vertex set J and such that a finite subset $\sigma \subseteq J$ spans a simplex if and only if the intersection $X_\sigma = \bigcap_{\alpha \in \sigma} X_\alpha$ is nonempty.

Lemma (Acyclic Covering Lemma). Suppose a CW complex X is a union of subcomplexes X_α such that every non-empty intersection $X_{\alpha_0} \cap X_{\alpha_1} \cap \cdots \cap X_{\alpha_p}$ is acyclic. Then $H_*(X) \cong H_*(N)$, where N is the nerve of the cover.

Exercise 40. (Bonus). Use the Mayer–Vietoris spectral sequence to prove the Acyclic Covering Lemma. See Brown [Br, Section VII.4].

Let R be a PID. Recall that an element $v = (r_1, r_2, \ldots, r_n) \in R^n$ is called *unimodular* if the ideal generated by $(r_1, r_2, \ldots, r_n)$ is R.

$$\mathscr{S}_q = \mathscr{S}_q(R^n) = \{(n+q) \times n \text{ matrices } A = (a_{i,j}) \text{ over } R \mid (a_{i,1}, \ldots, a_{i,n}) \text{ is unimodular for all } i\}$$
$$\mathscr{P}_q = \mathscr{P}_q(R^n) = \{A \in \mathscr{S}_q \mid \text{ each } n \times n \text{ submatrix has determinant } 0\}$$

These sets have actions of $\mathrm{GL}_n(R)$ by right multiplication. Let $C(\mathscr{S}_q)$ and $C(\mathscr{P}_q)$ denote the free abelian groups on $\mathscr{S}_q$ and $\mathscr{P}_q$, respectively, and let $C_q(R^n) = C(\mathscr{S}_q)/C(\mathscr{P}_q)$ be the quotient; it is a right $\mathbb{Z}[\mathrm{GL}_n(R)]$-module.

Our goal is to prove the following, Lee–Szczarba Theorem 3.1.

Theorem (The Sharbly resolution). There is an epimorphism $\phi : C_0(R^n) \to \mathrm{St}_n(R)$ of right $\mathbb{Z}[\mathrm{GL}_n(R)]$-modules so that

$$\longrightarrow C_q(R^n) \longrightarrow C_{q-1}(R^n) \longrightarrow \cdots \longrightarrow C_0(R^n) \xrightarrow{\phi} \mathrm{St}_n(R) \longrightarrow 0$$

is a free resolution of $\mathrm{St}_n(R)$ by $\mathbb{Z}[\mathrm{GL}_n(R)]$-modules.

Exercise 41. (a) Let K be the simplicial complex whose vertices are the unimodular elements of R^n, and whose simplices are all finite nonempty subsets of vertices. Show that K is contractible.

(b) Use the long exact sequence of a pair to show that, for a subcomplex $L \subseteq K$,

$$H_q(K, L) \cong \widetilde{H}_{q-1}(L).$$

(c) Let $L \subseteq K$ be the subcomplex consisting of all simplices with the property that all of their vertices lie in a proper direct summand of R^n. Let $\{H_i \mid i \in I\}$ be the set of direct summands of R^n of rank $(n-1)$. Let K_i be the full subcomplex of L with vertices lying in H_i. Show that $\{K_i \mid i \in I\}$ is an acyclic covering of L in the sense of the Acyclic Covering Lemma.
Hint: First argue that, since R is a PID, a nonempty intersection of summands H_i must be a direct summand of R^n isomorphic to R^r for some $0 < r < n$.

(d) Let N be the nerve of the cover $\{K_i \mid i \in I\}$. Use the Acyclic Covering Lemma to deduce that $H_q(L) \cong H_q(N)$ for all $q \geq 0$.

(e) Recall that $\mathbb{F}$ denotes the field of fractions of R. Let $\{W_j \mid j \in J\}$ denote the set of hyperplanes in $\mathbb{F}^n$. Let T_j be the subcomplex of the Tits building $\mathcal{T}_n(\mathbb{F})$ consisting of all simplices with W_j as a vertex. Show that $\{T_j \mid j \in J\}$ is an acyclic covering of L in the sense of the Acyclic Covering Lemma.

(f) Let $\tilde{N}$ be the nerve of the cover $\{T_j \mid j \in J\}$. Use the Acyclic Covering Lemma to deduce that $H_q(\mathcal{T}_n(\mathbb{F})) \cong H_q(\tilde{N})$ for all $q \geq 0$.

(g) Show that the mapping

$$\begin{aligned} \{\text{summands of } R^n\} &\longrightarrow \{\text{subspaces of } \mathbb{F}^n \cong R^n \otimes_R \mathbb{F}\} \\ H &\longmapsto H \otimes_R \mathbb{F} \end{aligned}$$

defines a simplicial isomorphism of N onto $\tilde{N}$.

(h) We have proved the following isomorphisms. Verify that they are $\mathrm{GL}_n(R)$-equivariant.

$$\begin{aligned} H_q(K, L) &\cong \widetilde{H}_{q-1}(L) \\ &\cong \widetilde{H}_{q-1}(N) \\ &\cong \widetilde{H}_{q-1}(\tilde{N}) \\ &\cong \widetilde{H}_{q-1}(\mathcal{T}_n(\mathbb{F})) \end{aligned}$$

(i) Verify that the $(n-2)$-skeleton of L coincides with the $(n-2)$-skeleton of K, so

$$C_q(K, L) = 0 \qquad \text{for } q \leq n-2.$$

(j) Using the Solomon–Tits result that $\mathcal{T}_n(\mathbb{F}) \simeq \bigvee S^{n-2}$, deduce that there is an exact sequence

$$\cdots \longrightarrow C_{q+n}(K, L) \longrightarrow C_{q+n-1}(K, L) \longrightarrow \cdots \longrightarrow C_{n-1}(K, L) \longrightarrow \mathrm{St}_n(R) \longrightarrow 0.$$

(k) Show that there are $\mathrm{GL}_n(R)$-equivariant isomorphisms of chain complexes

$$\begin{aligned} C_{q+n-1}(K) &\cong C(\mathscr{S}_q) \\ C_{q+n-1}(L) &\cong C(\mathscr{P}_q) \\ C_{q+n-1}(K, L) &\cong C_q(R^n) \end{aligned}$$

(l) Conclude the existence of the Sharbly resolution.

(m) Show that differentials in the Sharbly resolution are induced by the map

$$\begin{aligned} \mathscr{S}_q &\longrightarrow \mathscr{S}_{q-1} \\ A &\longmapsto \sum (-1)^i d_i(A) \end{aligned}$$

where d_i is the map that deletes the i^{th} row of the matrix A.

Prove that the Sharbly resolution is free.
Hint: First verify that $\mathrm{GL}_n(R)$ acts freely on the set of $(n+q) \times n$ matrices A satisfying

- each row of A is unimodular,
- some $(n \times n)$-submatrix of A has non-vanishing determinant.

Give a geometric interpretation of the map $\phi : C_0(R^n) \to \mathrm{St}_n(R)$. Conclude that the Steinberg module is generated by apartment classes.

Let R be a Euclidean ring with a multiplicative Euclidean norm. Using the Sharbly resolution, Lee–Szczarba went on to prove that the $\mathrm{SL}_n(R)$–coinvariants of $C_0(R^n)$ vanish, which implies that

$$H_0(\mathrm{SL}_n(R); \mathrm{St}_n(R)) = 0.$$

By virtual Bieri–Eckmann duality, this then implies that the rational cohomology of $\mathrm{SL}_n(R)$ vanishes in its virtual cohomological dimension.

Exercise 42.

(Bonus). Let R be a Euclidean ring with a multiplicative Euclidean norm. Prove directly that $\mathrm{SL}_n(R)$–coinvariants of $C_0(R^n)$ vanish. See Lee–Szczarba [LS, Theorem 4.1].

8.6 Maazen's theorem for $n = 2$

The exercises in this section are a warm-up to Wilson Lecture 3.

Definition (Complex of partial bases). Fix a PID R. Let $PB_n(R)$ denote the *complex of partial bases* in R^n. Its vertices are primitive elements v_0 of R^n, and vertices $\{v_0, \ldots, v_p\}$ span a p-simplex precisely when they are a subset of a basis for R^n (possibly equal to a basis of R^n).

Exercise 43. Fix a PID R. Let $P_n(R)$ denote the poset of partial bases of R^n under inclusion. Show that the geometric realization $|P_n(R)|$ is equal to the barycentric subdivision of $PB_n(R)$.

The goal of this section is to prove the following result of Maazen in the case $n = 2$.

Theorem (Maazen). Let R be a Euclidean domain. Let $P_n(R)$ denote the poset of partial bases of R^n under inclusion. Then $|P_n(R)|$ is Cohen–Macaulay.

Exercise 44. Fix a Euclidean ring R with norm $|\cdot|$, and let $n = 2$.

(a) Explain why, to prove Maazen's theorem in the case $n = 2$, it suffices to show that the graph $|P_2(R)|$ is connected, equivalently, that $PB_2(R)$ is connected. Thus given a vertex indexed by a primitive vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in R^2, it suffices to find a path to the vertex $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) Prove the following claim: Given a primitive vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in R^2 with $|b| > 0$, there is a basis

$$\left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\}$$

of R^2 with $|d| < |b|$.

Hint: First choose any partial basis $\left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c' \\ d' \end{bmatrix} \right\}$, and consider elements $\begin{bmatrix} c \\ d \end{bmatrix}$ of the form

$$\begin{bmatrix} c' \\ d' \end{bmatrix} - q \begin{bmatrix} a \\ b \end{bmatrix}, \qquad q \in R.$$

(c) Explain why the claim completes the proof.

(d) **(Bonus)**. Can you generalize this proof strategy to $n \geq 2$?

8.7 Poset homology

These exercises are meant to be attempted while or after reading the appendix on the topic.

Exercise 45. Let X be a poset and denote the simplicial complex of chains (or flags) in X by $|X|$. Show that the simplicial homology of $|X|$ agrees with the poset homology of X. (Use trivial coefficients for both, that means the functor $X \to \mathsf{Ab}$ that sends every object to $\mathbb{Z}$ and every map (inclusion) to the identity map on $\mathbb{Z}$.)

Exercise 46. Let K be a simplicial complex and X the poset of its simplices (ordered by inclusion). Check that $|X|$ is the barycentric subdivision of K.

Exercise 47. (harder) Prove the Solomon–Tits Theorem by induction using the reduced map-of-poset spectral sequence on the poset inclusion $\mathcal{T}_{n-1}(F) \subset \mathcal{T}_n(F)$.
Bonus: Show that apartments associated to bases of F^n that come from unipotent upper triangular matrices give a $\mathbb{Z}$-basis of $\mathrm{St}_n(F)$.

A Poset homology and the map of poset spectral sequence

Let X be a poset. We consider X as a category with its elements being objects and there is a morphism $x_0 \to x_1$ for elements $x_0, x_1 \in X$ exactly if $x_0 \leq x_1$. Let $F\colon X \to \mathsf{Ab}$ be a functor. Then we can define a chain complex

$$C_p(X;F) = \bigoplus_{x_0 < \cdots < x_p} F(x_0)$$

with the differential being the alternating sum of $d_i\colon C_p \to C_{p-1}$ where d_0 is given by the map $F(x_0) \to F(x_1)$ induced by $x_0 < x_1$ of the summands corresponding to $x_0 < \cdots < x_p$ and $x_1 < \cdots < x_p$, respectively, and for $i > 0$, d_i is the identity map $F(x_0) \to F(x_0)$ of the summands corresponding to $x_0 < \cdots < x_p$ and $x_0 < \cdots < \hat{x}_i < \cdots < x_p$. We define poset homology as the homology of this chain complex

$$H_p(X;F) = H_p(C_*(X;F)).$$

A map of posets $f\colon X \to Y$ is a functor between the categories, or concretely is a map such that $f(x_0) \leq f(x_1)$ in Y if $x_0 \leq x_1$ in X. This defines the category of posets. Define the poset fiber

$$f_{\leq y} = \{x \in X \mid f(x) \leq y\}.$$

Note that $y \mapsto H_p(f_{\leq y}; F)$ is a functor $Y \to \mathsf{Ab}$ because for $y_0 < y_1$ there is an inclusion

$$\bigoplus_{x_0 < \cdots < x_p; f(x_p) \leq y_0} F(x_0) \longrightarrow \bigoplus_{x_0 < \cdots < x_p; f(x_p) \leq y_1} F(x_0).$$

Map of poset spectral sequence: Let $f\colon X \to Y$ be a map of posets. There is a spectral sequence

$$E^2_{pq} = H_p(Y; y \mapsto H_q(f_{\leq y}; F)) \implies H_{p+q}(X;F).$$

Proof. Consider the double complex

$$D_{pq} = \bigoplus_{\substack{x_0 < \cdots < x_p \\ y_0 < \cdots < y_q \\ f(x_p) \le y_0}} F(x_0).$$

The corresponding spectral sequence is

$$E^1_{pq} \cong \bigoplus_{x_0 < \cdots < x_p} H_q(Y_{\ge f(x_p)}; F(x_0))$$

on the first page. Because $Y_{\ge f(x_p)}$ has a cone point in $f(x_p)$, it is contractible and this spectral sequence actually simplifies to

$$E^1_{pq} \cong \begin{cases} \bigoplus_{x_0 < \cdots < x_p} F(x_0) & q = 0 \\ 0 & q \neq 0. \end{cases}$$

Therefore the second page collapses to

$$E^2_{pq} \cong \begin{cases} H_p(X; F) & q = 0 \\ 0 & q \neq 0. \end{cases}$$

And this implies that the spectral sequence converges to $H_{p+q}(X; F)$.

Now transposing the double complex, we get another spectral sequence that also converges to $H_{p+q}(X; F)$. This spectral sequence is

$$E^1_{pq} \cong \bigoplus_{y_0 < \cdots < y_p} H_q(f_{\le y_0}; F)$$

and thus

$$E^2_{pq} \cong H_p(Y; y \mapsto H_q(f_{\le y}; F))$$

as asserted. □

With constant integer coefficients, we may define reduced poset homology by augmenting the chain complex to

$$\tilde{C}_p(X; \mathbb{Z}) = \begin{cases} \mathbb{Z} & p = -1, \\ \displaystyle\bigoplus_{x_0 < \cdots < x_p} \mathbb{Z} & p \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

More generally, we also define the relative homology $H_*(f)$ of a map of posets $f\colon X \to Y$ to be the homology of the mapping cone of $C_*(X;\mathbb{Z}) \to C_*(Y;\mathbb{Z})$.

Reduced map of poset spectral sequences: Let $f\colon X \to Y$ be a map of posets. There is a spectral sequence

$$E^2_{pq} = H_p(Y; y \mapsto \widetilde{H}_q(f_{\leq y};\mathbb{Z})) \implies H_{p+q+1}(f).$$

Proof. Let us consider the double complex

$$D_{pq} = \begin{cases} \displaystyle\bigoplus_{y_0 < \cdots < y_q} \mathbb{Z} & p = -1, q \geq 0, \\ \displaystyle\bigoplus_{\substack{x_0 < \cdots < x_p \\ y_0 < \cdots < y_q \\ f(x_p) \leq y_0}} \mathbb{Z} & p, q \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding spectral sequence is

$$E^1_{pq} \cong \begin{cases} H_q(Y;\mathbb{Z}) & p = -1, q \geq 0, \\ \displaystyle\bigoplus_{x_0 < \cdots < x_p} H_q(Y_{\geq f(x_p)};\mathbb{Z}) & p, q \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Because $Y_{\geq f(x_p)}$ is contractible, $E^1_{pq} \cong 0$ unless $p = -1$ or $q = 0$. Turning all the pages, provides the following long exact sequence for the total homology of the double complex.

$$\cdots \to H_1(X;\mathbb{Z}) \to H_1(Y;\mathbb{Z}) \to H_0(Tot_*(D)) \to H_0(X;\mathbb{Z}) \to H_0(Y;\mathbb{Z}) \to H_{-1}(Tot_*(D)) \to 0$$

This implies that the spectral sequence converges to $H_{p+q+1}(f)$.

Now transposing the double complex, we get another spectral sequence that also converges to $H_{p+q+1}(f)$. This spectral sequence is

$$E^1_{pq} \cong \bigoplus_{y_0 < \cdots < y_p} \widetilde{H}_q(f_{\leq y_0};\mathbb{Z})$$

and thus

$$E^2_{pq} \cong H_p(Y; y \mapsto \tilde{H}_q(f_{\leq y};\mathbb{Z}))$$

as asserted. □

Functors supported on an antichain: Suppose that $F\colon \mathbf{Y} \to \mathsf{Ab}$ is a functor supported on elements of an anitchain A. Then

$$H_p(Y;F) = \bigoplus_{y_0 \in A} \widetilde{H}_{p-1}(Y_{>y_0}; F(y_0)).$$

Proof. Suppose that $F\colon \mathbf{Y} \to \mathsf{Ab}$ is supported on elements of an antichain A.

$$C_p(Y;F) = \bigoplus_{y_0<\cdots<y_p\in Y} F(y_0) \cong \bigoplus_{y_0\in A}\left(F(y_0)\otimes_{\mathbb{Z}} \bigoplus_{y_0<\cdots<y_p}\mathbb{Z}\right)$$
$$\cong \bigoplus_{y_0\in A}\Big(F(y_0)\otimes_{\mathbb{Z}}\widetilde{C}_{p-1}(Y_{>y_0};\mathbb{Z})\Big).$$

The composition of these isomorphisms is compatible with the differentials because $d_0 = 0$ as no $F(y_1) = 0$ for all $y_1 > y_0$. (The support of F would not be an anitchain otherwise.) Hence gives an isomorphism of chain complexes. Thus

$$H_p(Y;F) = \bigoplus_{y_0\in A}\widetilde{H}_{p-1}(Y_{>y_0};F(y_0)). \qquad \square$$

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Please send comments & corrections!
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