Classifying Surfaces

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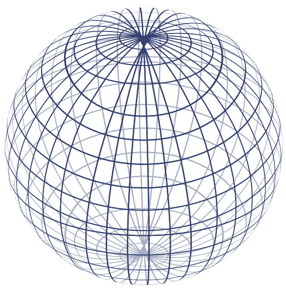
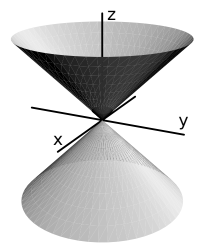
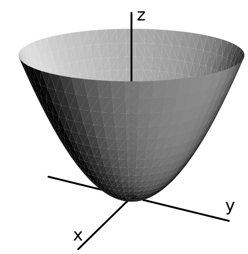
Exercise # 1. (Recognizing surfaces) Which of the following are surfaces? Which are surfaces with boundary?

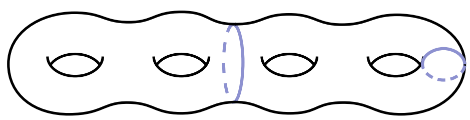

(a) 2–Sphere S^2

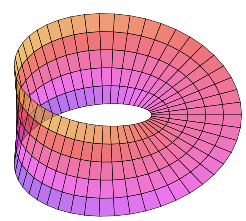


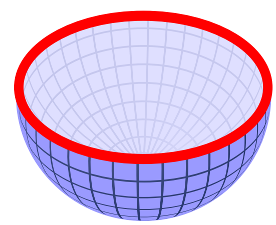
(b) Solution to $x^2 + y^2 = z^2$

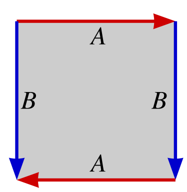


(c) Solution to $z = x^2 + y^2$

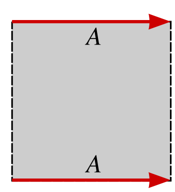

(d) Genus 4 Surface


(e) Closed Mobius Strip

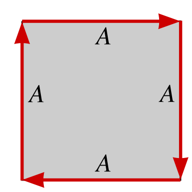

(f) Closed Hemisphere



(g) Quotient Space $ABAB^{-1}$

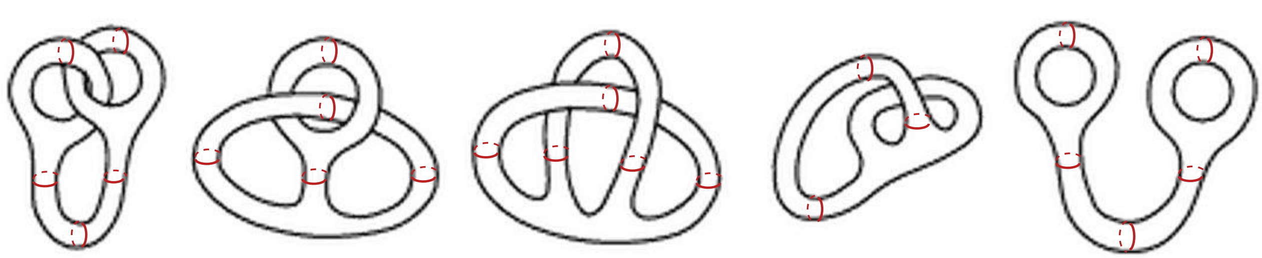


(h) Quotient Space AA

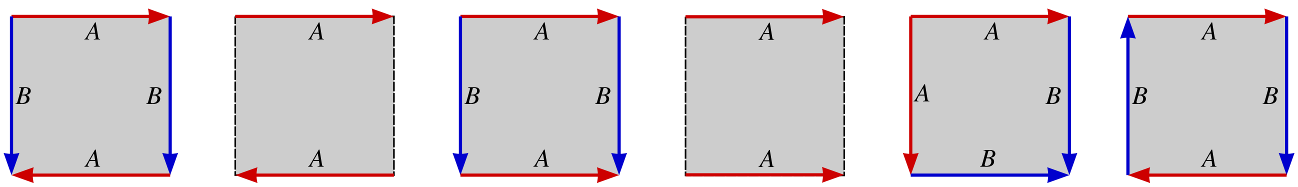


(i) Quotient Space $AAAA$

Exercise # 2. (Isomorphisms of surfaces) Sort the following surfaces into isomorphism classes.

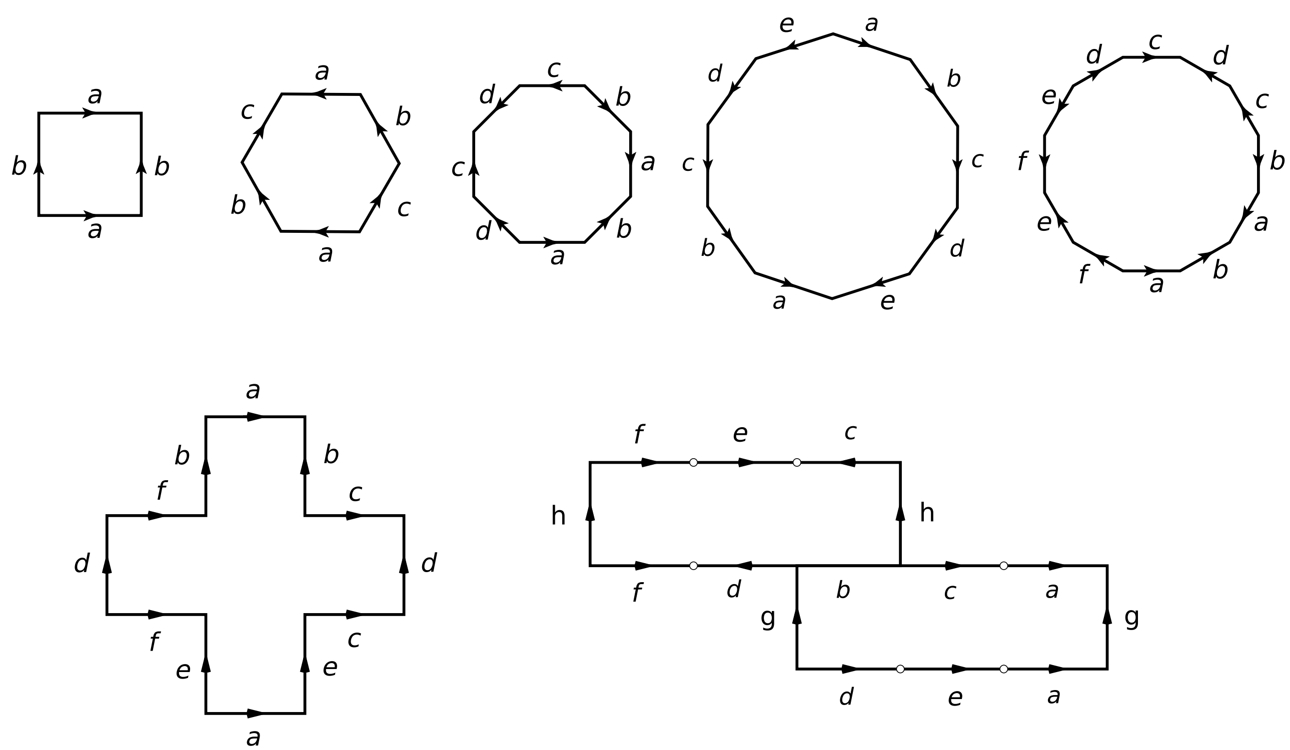


Exercise # 3. (Quotient surfaces) Identify among the following quotient spaces: a cylinder, a Möbius band, a sphere, a torus, real projective space, and a Klein bottle.



Exercise # 4. (Gluing Mobius bands) How many boundary components does a Mobius band have? What surface do you get by gluing two Mobius bands along their boundary components?

Exercise # 5. (More quotient surfaces) Identify the following surfaces.



Triangulations

Exercise # 6. (Minimal triangulations) What is the minimum number of triangles needed to triangulate a sphere? A cylinder? A torus?

Orientability

Exercise # 7. (Orientability is well-defined) Fix a surface S. Prove that if one triangulation of S is orientable, then all triangulations of S are orientable.

Exercise # 8. (Orientability) Prove that the disk, sphere, and torus are orientable. Prove that the Mobius strip and Klein bottle are nonorientiable. Is a genus g surface S_g always orientable?

Euler Characteristic

Exercise # 9. (Euler characteristic of a closed 2–disk $\overline{D^2}$) Given any triangulation of a closed disk $\overline{D^2}$, prove that

$$\chi := \#\text{Vertices} - \#\text{Edges} + \#\text{Faces} \qquad \text{is always equal to } 1.$$

Hint: First prove this is true for a single triangle, then proceed by induction on the number of triangles. What happens to the alternating sum χ when you delete a triangle?

Exercise # 10. (Euler characteristic of a 2–sphere S^2) Use the result of Exercise #9 to prove that for any triangulation of a 2–sphere S^2, $\chi = 2$.

Exercise # 11. (More Euler characteristics) For each of the surfaces in Exercises #3 and #5, choose a triangulation and use it to compute the Euler characteristic of the surface.