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Cohomology of Braid Groups & Configuration Spaces

Configuration Space

Let X be a top space (we'll assume connected).

Defn $\mathcal{F}_n(X) = \{ (x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j \}$

is the ordered n-point configuration space of X.

$$\mathcal{F}_n(x) = X^n - \Delta \hookrightarrow X^n$$

"fat diagonal" – all n-tuples with some repeated coordinate.

Symmetric gp $S_n \curvearrowright \mathcal{F}_n(X)$
permutes coordinates

Gives covering map

$$\begin{array}{c} \mathcal{F}_n(X) \\ \downarrow \\ B_n(X) := \mathcal{F}_n(X) / S_n \end{array}$$

(unordered) n-point configuration space of X.

Our goal: Study $H^*(\mathcal{F}_n(X))$ and $H^*(B_n(X))$, X an oriented manifold

Fact (Transfer map) $H^p(\mathcal{F}_n(X); \mathbb{Q})^{S_n} \cong H^p(B_n(X); \mathbb{Q})$

S_n-invariants

Idea: Average over action of Deck group S_n.

So it suffices to understand $H^*(\mathcal{F}_n(X))$ as an S_n-rep.

Key tool to study:

There is a map "forget a point" $\quad \mathcal{F}_{n+1}(X) \longrightarrow \mathcal{F}_n(X)$

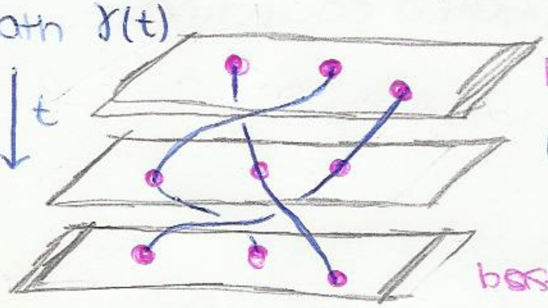
$$(x_1, \dots, x_n, x_{n+1}) \longmapsto (x_1, \dots, x_n).$$

<u>NB</u> There is <u>no</u> corresponding map $B_{n+1}(X) \longrightarrow B_n(X)$ since it is generally not possible to continuously choose which point to forget among $(n+1)$ unordered points.

Warm-up case: $\mathcal{F}_n(\mathbb{C})$

<u>Fact:</u> $\pi_1(B_n(\mathbb{C})) = B_n$, Artin's braid gp.

path $\gamma(t)$

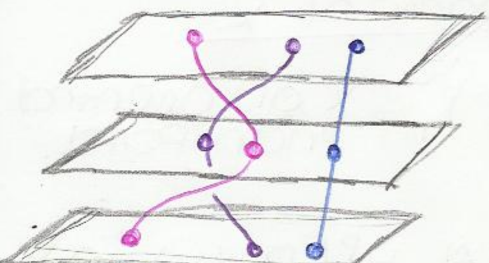


basept $x \in B_n(\mathbb{C})$ $\gamma(0) = x$

Loop in $B_n(\mathbb{C})$ corresponds to a braid.

basept $x \in B_n(\mathbb{C})$. $\gamma(1) = x$

$\pi_1(\mathcal{F}_n(\mathbb{C})) = P_n$, the pure (coloured) braid gp



$P_n \leq B_n$ is the subgroup of braids where each strand begins and ends in the same position.

The quotient is S_n:

$$\begin{array}{c} \mathcal{F}_n(\mathbb{C}) \circlearrowleft B_n/P_n = S_n \\ \downarrow \\ B_n(\mathbb{C}) \end{array}$$

$$1 \longrightarrow P_n \longrightarrow B_n \longrightarrow S_n \longrightarrow 1.$$

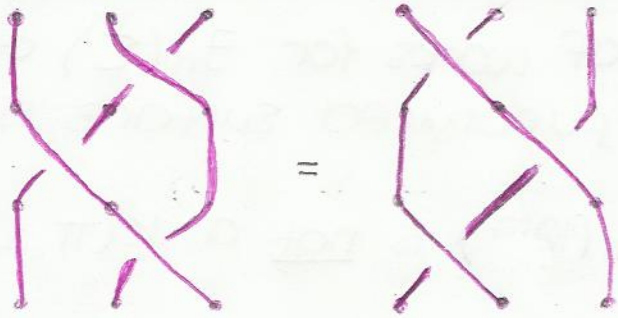
Group structure

Generators s_i

i over i+1 strand.

$$B_n = \langle s_1, \dots, s_{n-1} \mid \underbrace{s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}}_{\uparrow \text{ Braid relation}},\ s_i s_j = s_j s_i \text{ for } |i-j| > 1 \rangle$$

Braid relation:



(pull frontmost strand left to right)

The generators s_i are not elements of P_n.

P_n generators $T_{ij} = T_{ji}$

wrap ith strand once counterclockwise around jth (equivalently, jth once counterclockwise around ith)

$P_n = \langle T_{ij},\ i \neq j \in [n] \mid$ Finitely many relations, all contained in $[P_n, P_n] \rangle$.

<u>Prop</u> $\mathcal{F}_n(\mathbb{C})$ and $B_n(\mathbb{C})$ are $K(\pi, 1)$'s

- Suffices to show their universal cover is contractible, by showing $\pi_i(\mathcal{F}_n(\mathbb{C})) = 0$ for $i > 1$.

<u>Proof Idea:</u> Fibration

$$\begin{array}{ccc} F & \longrightarrow & \mathcal{F}_{n+1}(\mathbb{C}) \\ \| & & \downarrow \\ \mathbb{C} \setminus \{n \text{ points}\} & & \mathcal{F}_n(\mathbb{C}) \\ \simeq \bigvee_n S^1 & & \end{array}$$

Extension:

$$F_n \longrightarrow P_{n+1} \longrightarrow P_n$$

Gives LES on homotopy gps. For $i > 1$:

$$\longrightarrow \underset{\substack{\| \\ 0}}{\pi_{i+1}(F)} \longrightarrow \pi_{i+1}(\mathcal{F}_{n+1}(\mathbb{C})) \xrightarrow{\simeq} \pi_{i+1}(\mathcal{F}_n(\mathbb{C})) \longrightarrow \underset{\substack{\| \\ 0}}{\pi_i(F)} \longrightarrow$$

After checking base cases ($n = 1$, $\mathcal{F}_1(\mathbb{C}) \cong \mathbb{C}$) and $i = 1$, can show $\mathcal{F}_n(\mathbb{C})$ is aspherical by induction on n and i.

NB This proof works for $\mathcal{F}_n(\Sigma)$ of a surface Σ, since a punctured surface is aspherical.

However, $\mathcal{F}_n(\mathbb{R}^m)$ is not a $K(\pi,1)$ for $m>2$.

$$\begin{array}{ccc} \mathbb{R}^m \setminus \{n \text{ points}\} & \longrightarrow & \mathcal{F}_{n+1}(\mathbb{R}^m) \\ \simeq & & \downarrow \\ \bigvee_n S^{m-1} & & \mathcal{F}_n(\mathbb{R}^m) \end{array}$$

$\uparrow$ not aspherical.

In fact, $\pi_1(\mathcal{F}_n(\mathbb{R}^m)) = 0$ for $m>2$, since, given a loop in $\mathcal{F}_n(\mathbb{R}^m)$, there is "enough room" in $\mathbb{R}^m$ to unbraid it.

A section.

The fibration $\mathcal{F}_{n+1}(\mathbb{C}) \longrightarrow \mathcal{F}_n(\mathbb{C})$ has a section

$$(x_1, \dots, x_n, x_{n+1}) \longmapsto (x_1, \dots, x_n)$$

$$(x_1, \dots, x_n, \max_i |x_i| + 1) \longleftarrow (x_1, \dots, x_n)$$

more generally, $\mathcal{F}_{n+1}(X) \longrightarrow \mathcal{F}_n(X)$ has a section for open manifolds X by "adding points at infinity".

This section will feature in our computation of $H^*(P_n)$.

Computing $H^*(P_n;\mathbb{Z}) = H^*(\mathcal{F}_n(\mathbb{C});\mathbb{Z})$

$H_1(P_n;\mathbb{Z}) = \mathbb{Z}[T_{ij}]$ abelianization of P_n is free abelian gp on T_{ij}

$H^1(P_n;\mathbb{Z}) = \mathbb{Z}[T_{ij}^*]$

$H^*(P_n;\mathbb{Z}) = \langle T_{ij}^*\rangle$ we will see that H^* is generated as a ring by dual elements T_{ij}^*

We can identify T_{ij}^* with form on $\mathcal{F}_n(\mathbb{C})$

$$w_{ij} := \frac{1}{2\pi i}\,\frac{dz_i - dz_j}{z_i - z_j}$$

computes "winding number" of a loop around the missing diagonal $z_i = z_j$.

As a function on pure braids, T_{ij}^* counts # times (with sign) the ith strand winds around the jth.

<u>Thm</u> (Arnold, 1968) $H^*(P_n;\mathbb{Z}) \cong \Lambda\langle w_{ij}\rangle / \langle R_{ijk}\rangle$ (exterior algebra)

where R_{ijk} is the relation

$$w_{ij}w_{jk} + w_{jk}w_{ki} + w_{ki}w_{ij} = 0$$

Consequences:

- $H^*(P_n;\mathbb{Z})$ is torsion free
- H^p has additive basis

$$B = \{ w_{k_1 l_1} \cdots w_{k_p l_p} \mid k_s < l_s,\ l_s < l_{s+1} \}$$

Need to verify: Relation R_{ijk} implies generated by words with <u>strictly</u> decreasing second index.

- The Poincaré polynomial (generating function for the Betti numbers)

is $p(t) = (1+t)(1+2t)\cdots(1+(n-1)t)$

Outline of Pf

① Check by hand that the forms ω_{ij} satisfy R_{ijk}.
(this is basic arithmetic)
This gives a ring map $\Lambda\langle \omega_{ij}\rangle / \langle R_{ijk}\rangle \longrightarrow H^* P_n$

② Use the relations R_{ijk} to show B generates $\Lambda\langle \omega_{ij}\rangle / \langle R_{ijk}\rangle$. (routine algebra)

③ Use the Leray-Serre spectral sequence for the fibration $\mathcal{F}_{n+1}(\mathbb{C}) \longrightarrow \mathcal{F}_n(\mathbb{C})$ to compute $H^*(P_n)$ as a $\mathbb{Z}$-module.

Deduce from this computation that the elements of B are a linear basis for $H^p(P_n)$.

hence $\Lambda\langle \omega_{ij}\rangle / R_{ijk} \xrightarrow{\cong} H^*(P_n)$

Steps ① and ② are left as (not so hard) exercises, and we will expand on ③.

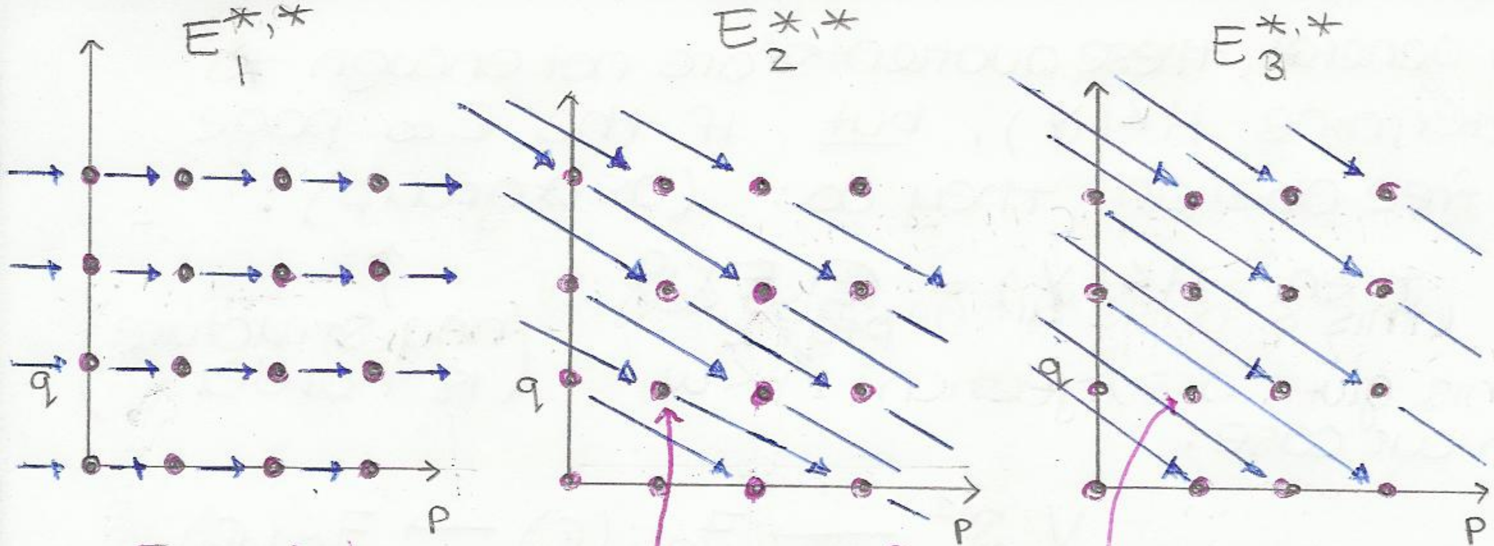
The Leray-Serre Spectral Sequence

Given a fibration $F \longrightarrow Y \longrightarrow X$ (X assumed path-connected)
there is a spectral sequence

$$E_2^{p,q} = H^p(X; H^q(F)) \Longrightarrow E_\infty^{p,q} = H^{p+q}(Y;\mathbb{Z}).$$

Recall what this means:

For each $r \geq 2$, the page $E_r^{p,q}$ is a bigraded $\mathbb{Z}$-module



Each dot represents a $\mathbb{Z}$-module $E_2^{p,q} \cong H^p(X; H^q(F))$

$E_{r+1}^{p,q}$ is a subquotient of $E_r^{p,q}$
$E_{r+1}^{p,q} = \ker(d_r) / \mathrm{im}(d_r)$ at $E_r^{p,q}$.

Each page E_r^{**} has a differential map $d_r : E_r^{**} \circlearrowleft$ satisfying $(d_r)^2 = 0$

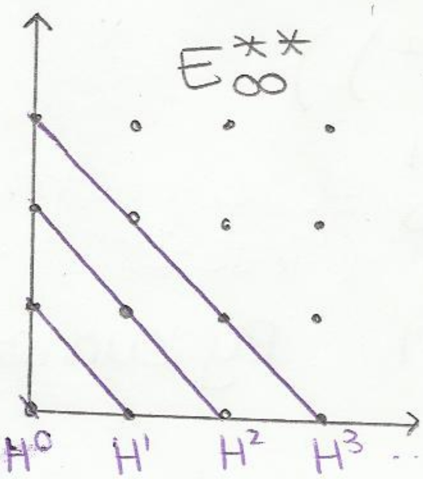
The groups E_{r+1}^{**} are the homology of (E_r^{**}, d_r)

For the Leray-Serre spectral sequence

- $d_r : E_r^{p,q} \longrightarrow E_r^{p+r,\, q-r+1}$
- $E_r^{p,q} = 0$ outside the first quadrant

This means, eventually for each p,q, the incoming and outgoing differential must have domain or range 0 (outside first quadrant)

and so $E_r^{p,q}$ stabilizes. This limit is called $E_\infty^{p,q}$.



In the limit, the groups on the diagonal $E_\infty^{p,k-p}$ are the associated graded pieces to a filtration

total space

$0 \subseteq F_k^k \subseteq \cdots \subseteq F_0^k = H^k(Y; \mathbb{Z})$

ie, $E_\infty^{p,k-p} = F_p^k / F_{p+1}^k$

In general, these quotients are not enough to determine $H^k(Y)$, <u>but</u>, if the E_∞ page is free abelian, they do (as a group):

then $$H^k(Y) = \bigoplus_{p+q=k} E_\infty^{p,q}$$

(ring structure is harder)

In our case:

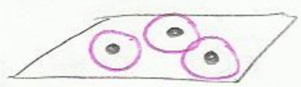
$$\bigvee_n S^1 \longrightarrow \mathcal{F}_{n+1}(\mathbb{C}) \longrightarrow \mathcal{F}_n(\mathbb{C})$$

- our base $\mathcal{F}_n(\mathbb{C})$ is path-connected

  (any n-tuple of points can be smoothly deformed to any other without points colliding)

- The action of $\pi_1(\mathcal{F}_n(\mathbb{C})) = P_n \curvearrowright H^*(V_n S^1)$ is trivial:

  the cohomology is generated by loops around the punctures — each puncture returns to its starting position after tracing out a pure braid



This means the coefficients $H^q(F)$ in
$$E_2^{p,q} = H^p(X; H^q(F))$$
are the trivial $\pi_1(X)$-module, and we do not need to contend with "twisted" coefficients.

- $$E_2^{p,q} = H^p(\mathcal{F}_n(\mathbb{C}); H^q(V_n S^1))$$
$$= \begin{cases} H^p(\mathcal{F}_n(\mathbb{C}); \mathbb{Z}^n), & q=1 \\ H^p(\mathcal{F}_n(\mathbb{C}); \mathbb{Z}), & q=0 \\ 0 & q>1 \end{cases}$$
$$= \begin{cases} H^p(\mathcal{F}_n(\mathbb{C})) \otimes \mathbb{Z}^n, & q=1 \\ H^p(\mathcal{F}_n(\mathbb{C})), & q=0 \\ 0, & q>1 \end{cases}$$ By Kunneth

2	0	0	0	0
1	$H^0(P_n)\otimes\mathbb{Z}^n$	$H^1(P_n)\otimes\mathbb{Z}^n$	$H^2(P_n)\otimes\mathbb{Z}^n$	$H^3(P_n)\otimes\mathbb{Z}^n$
0	$H^0(P_n)$	$H^1(P_n)$	$H^2(P_n)$	$H^3(P_n)$
	0	1	2	

The only possible nonzero differential is d_2^{**}; all others must have domain or codomain 0.

Fact: Because the fibration has a section, the differential d_2 is 0, and $E_2^{p,q} = E_\infty$.

(Explanation shortly)

<u>and</u> by induction, all bigraded pieces are free abelian, so

$$H^k(\mathcal{F}_{n+1}(\mathbb{C})) \cong H^k(\mathcal{F}_n(\mathbb{C})) \oplus H^{k-1}(\mathcal{F}_n(\mathbb{C}))\otimes\mathbb{Z}^n$$

The map $\vee_n S^1 \simeq F \xhookrightarrow{i} \mathcal{F}_{n+1}(\mathbb{C})$ induces

$$\mathbb{Z}^n \cong H^1(\vee_n S^1) \xleftarrow{i^*} H^1(\mathcal{F}_{n+1}(\mathbb{C})).$$

and we can identify $\mathbb{Z}^n \cong \mathbb{Z}[\omega_{1\,n+1}, \ldots, \omega_{n\,n+1}]$

By induction, the group $H^k(\mathcal{F}_{n+1}(\mathbb{C}))$ has $\mathcal{B}$ as its additive basis.

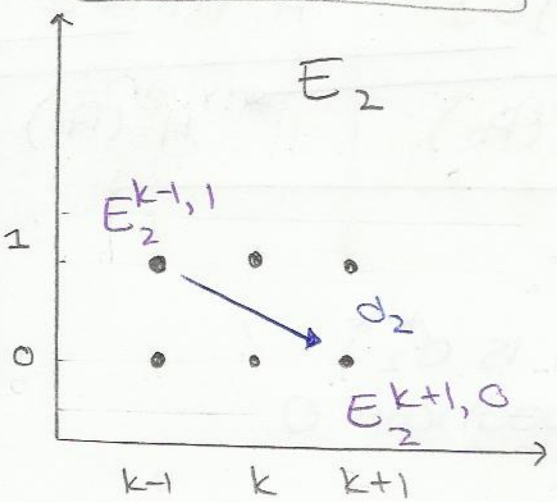
This gives our isomorphism

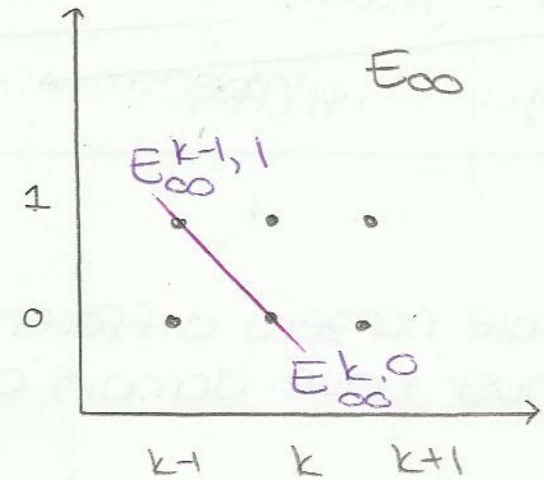
$$\Lambda[\omega_{ij}] / \langle R_{ijk} \rangle \xrightarrow{\cong} H^*(\mathcal{F}_n(\mathbb{C})).$$

Why does $d_2 \equiv 0$?

$$\boxed{F \longrightarrow E \xrightarrow[\pi]{} X}$$

The following argument may be familiar to those who have studied sphere bundles:





Since d_2 is the only possible nonzero differential,

$E_\infty^{k-1,1} = \ker(d_2)$ at $E_2^{k-1,1}$; $E_\infty^{k+1,0} = \dfrac{E_2^{k+1,0}}{\mathrm{Im}(d_2)}$

which gives an exact sequence

$$0 \hookrightarrow E_\infty^{k-1,1} \hookrightarrow E_2^{k-1,1} \xrightarrow{d_2} E_2^{k+1,0} \twoheadrightarrow E_\infty^{k+1} \longrightarrow 0$$

The cohomology $H^k(Y)$ is the limit of the sequence in the sense that it is an extension:

$$0 \longrightarrow E_\infty^{k,0} \longrightarrow H^k(Y, \mathbb{Z}) \longrightarrow E_\infty^{k-1,1} \longrightarrow 0$$

Combining these exact sequences gives a long exact sequence

$$\longrightarrow H^k(Y) \longrightarrow E_2^{k-1,1} \xrightarrow{d_2} \underset{\substack{\| \\ H^{k+1}(X)}}{E^{k+1,0}} \longrightarrow H^{k+1}(Y) \longrightarrow$$

This is analogous to the Gysin sequence of a sphere bundle

The map $H^{k+1}(X) \longrightarrow H^{k+1}(Y)$ is precisely the map π^* induced by the fibration $\pi : Y \longrightarrow X$.

Since π has a section, π^* injects, and so by exactness $d_2 \equiv 0$.

Remark: $\mathcal{F}_n(\mathbb{R}^m)$

Note that the same spectral sequence argument given by Arnold can be used to compute the groups $H^k(\mathcal{F}_n(\mathbb{R}^m), \mathbb{Z})$:

$E_2 = E_\infty$

$$E_2^{p,q} = \begin{cases} H^p(\mathcal{F}_n \mathbb{R}^m), & q=0 \\ H^p(\mathcal{F}_n \mathbb{R}^m) \otimes \mathbb{Z}^n, & q=m-1 \\ 0 & \text{ow} \end{cases}$$

$$\begin{array}{c|ccc}
m-1 & H^0(\mathcal{F}_n \mathbb{R}^m) \otimes \mathbb{Z}^n & H^1(\mathcal{F}_n \mathbb{R}^m) \otimes \mathbb{Z}^n & \cdots \\
 & 0 & 0 & \\
 & \vdots & & \\
 & 0 & 0 & \\
0 & H^0(\mathcal{F}_n \mathbb{R}^m) & H^1(\mathcal{F}_n \mathbb{R}^m) & \cdots \\
\hline
\end{array}$$

and again by induction the group

$$H^{k(m-1)}(\mathcal{F}_n(\mathbb{R}^m); \mathbb{Z}) \cong \mathbb{Z}[G_{a_1 b_1} \cdots G_{a_k b_k} \mid a_i < b_i,\ b_s < b_{s+1}]$$

where $|G_{ab}| = (m-1)$.

F. Cohen computed the ring $H^*(\mathcal{F}_n(\mathbb{R}^m))$ (1988)

It is the skew-commutative graded algebra on degree $(m-1)$-generators G_{ab} $\left(\begin{smallmatrix} a \neq b \\ \in [n] \end{smallmatrix}\right)$ $G_{ba} = (-1)^m G_{ab}$ modulo $(G_{ab})^2 = 0$, $G_{ab}G_{ac} + G_{bc}G_{ba} + G_{ca}G_{cb} = 0$ with S_n-action $\sigma : G_{ab} \longmapsto G_{\sigma(a)\sigma(b)}$

Totaro & Configuration Spaces of manifolds:

Totaro (1993) studied the cohomology $H^*(\mathcal{F}_n(X); \mathbb{Q})$ for X a closed real oriented manifold — by realizing the cohomology as the limit of a Leray spectral sequence. When X has the additional structure of a complex projective variety, Totaro proves that the sequence degenerates after the first nonzero differential.

Totaro's work is motivated in part by the question:

<u>Q</u> Is $H^*(\mathcal{F}_n(X); R)$ determined by $H^*(X; R)$?

<u>A</u> No — in general, as groups.

Totaro exhibits two $\mathbb{CP}^1$-bundles over $\mathbb{CP}^2$, X and Y, so that X and Y have the same rational Betti numbers, but $B_n(X)$ and $B_n(Y)$ do not.

<u>But</u> Totaro shows that if X is a smooth complex projective variety, the ring $H^*(X; \mathbb{Q})$ determines $H^*(\mathcal{F}_n(X); \mathbb{Q})$.

The Leray Spectral Sequence

Let X be a closed oriented manifold, $\dim_{\mathbb{R}} X = m$.

the inclusion $\mathcal{F}_n(X) \xrightarrow{f} X^n$

gives rise to a Leray spectral sequence

$$E_2^{p,q} = H^p(X^n; R^q f_* \mathbb{Z}) \Rightarrow E_\infty^{p,q} = H^{p+q}(\mathcal{F}_n(X); \mathbb{Z})$$

where $R^q f_* \mathbb{Z}$ is the sheaf associated to the presheaf

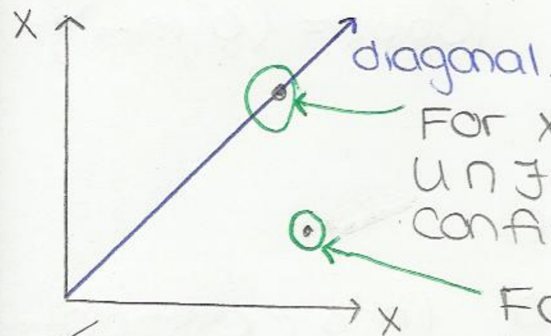
$$X_n \supseteq U \longmapsto H^q(U \cap \mathcal{F}_n(X); \mathbb{Z}).$$

The key insight to the computation is that this sheaf is locally constant, and supported on diagonals of X^n.

More specifically, given any point $x \in X^n$, the map

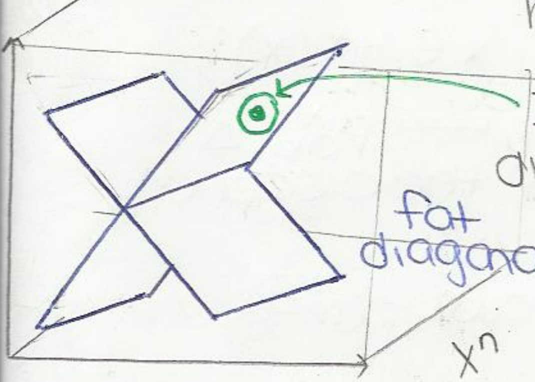
$$U \longmapsto H^q(U \cap \mathcal{F}_n(X); \mathbb{Z})$$

is constant on sufficiently small balls around x.



For x on a diagonal, we can relate $U \cap \mathcal{F}_n(X)$ back to a product of Euclidean configurations -

For x in $\mathcal{F}_n(X)$, $U \cap \mathcal{F}_n(X) = U$ is homologically trivial.



In general, for any point on the diagonal, we associate a partition of n

$$(\underbrace{y,y,y}_{\updownarrow},\underbrace{z,z}_{\updownarrow},\underbrace{w,w}_{\updownarrow}) \mapsto (3,2,2)$$

Identify nbhd with product

$$\mathcal{F}_3(\mathbb{R}^m) \times \mathcal{F}_2(\mathbb{R}^m) \times \mathcal{F}_2(\mathbb{R}^m)$$

Choose Reim. metric and exponentiate to identify $U \cap X \leftrightarrow \mathbb{R}^m$

and the cohomology groups of products $\mathcal{F}_i(\mathbb{R}^m)$ follow from Cohen's computation and Kunneth.

Totaro's result is:

Thm (Totaro)

Let X be a closed real oriented manifold
Let R be $\mathbb{Z}$ or a field; coefficients in R

Let $p_a^*\, H^*(X) \to H^*(X^n)$ and $p_{ab}^* : H^*(X^2) \to H^*(X^n)$ pull back the projections p_a, p_{ab} $(a \neq b \in [n])$.

Then the Leray spectral sequence for $F_n(X) \hookrightarrow X^n$ has E_2 page a quotient of the (skew-commutative) bigraded R-algebra

$$H^*(X^n; R)[G_{ab}] \qquad (1 \le a, b \le n,\ a \neq b)$$

degrees: $|x| = (i, 0)$, $x \in H^i$; $|G_{ab}| = (0, m-1)$

modulo

(0) $G_{ab} = (-1)^m G_{ba}$, (1) $(G_{ab})^2 = 0$

(2) $G_{ab}G_{ac} + G_{bc}G_{ba} + G_{ca}G_{cb} = 0$ (a, b, c distinct)

(3) $p_a^*(x)\, G_{ab} = p_b^*(x)\, G_{ab}$ $(a \neq b,\ x \in H^*(X))$.

The first nonzero differential $d_m : G_{ab} \longmapsto p_{ab}^* \Delta$ where $\Delta \in H^m(X^2)$ is the class of the diagonal.

- E_2: nonzero along rows $r(m-1)$, $0 \le r \le n-1$
- $E_2^{p, r(m-1)} = \bigoplus_J H^p(X_J^{n-r}) \otimes R^{c_J}$

$2(m-1)$	$\bigoplus_J H^0(X_J^{n-2}) \otimes R^{c_J}$	$\bigoplus_J H^1(X_J^{n-2}) \otimes R^{c_J}$
$m-1$	$\bigoplus_J H^0(X_J^{n-1})$	$\bigoplus_J H^1(X_J^{n-1})$
0	$H^0(X^n)$	$H^0(X^n)$

J: partition of n into $n-r$ blocks

$X_J^{n-r} \cong X^{n-r}$ diagonal associated to J.

For $J = (j_1, \dots, j_{n-r})$
$c_J = (j_1 - 1)! \cdots (j_{n-r} - 1)!$

Smooth complex projective varieties.

Totaro shows that, when X is complex projective variety, its Hodge filtration is compatible with the differential and the gradings in the Leray spectral sequence,

and group $E_2^{i,r(m-1)}$ is pure weight $(i+rm)$. $\boxed{m \text{ is } \underline{\text{real}} \text{ dim}}$

Since the differentials respect the weights, it follows that the only possible nonzero differential is d_m, and the limit of the sequence is the homology of the E_2 page $H(E_2^{**}, d_m)$

In this case, Totaro argues, the limit depends only on the cohomology $H^*(X)$ and the class of the diagonal in $H^*(X^2)$, which depends on the ring structure of $H^*(X)$.