

Constructing Spectral Sequences.

Goal: Constructing the spectral sequence associated to a filtered space  $\emptyset \subseteq X_0 \subseteq X_1 \subseteq \dots \subseteq X_n = X$

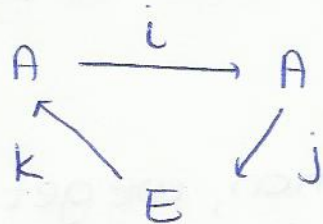
We will assume

- finite filtration
- spaces have homology groups of finite dimension/rank.

Intuition: Starting from the relative groups  $H_n(X_p, X_{p-1})$ , we will construct "successive approximations" to the homology  $H_n(X)$ .

Preliminaries: Exact Couples.

Defn An exact couple is an exact sequence of abelian groups of the form:



with  $i, j, k$  group homomorphisms.

Define map  $d := j \circ k : E \rightarrow E$ .

Claim  $d^2 = 0$

Pf  $d^2 = (jk)(jk) = (j)(\underbrace{kj})_k = 0$   
 • by exactness.

We will give a procedure for extending  $(E, d)$  to a spectral sequence.

Make the following definitions:

$$E^2 = \frac{\ker d}{\text{Im } d} \quad A^2 = i(A)$$

$$i^2: A^2 \longrightarrow A^2$$

$$i^2 = i \mid_{i(A)}$$

\*NB the superscripts are indices, and not exponents.

$$k^2: E^2 \longrightarrow A^2$$

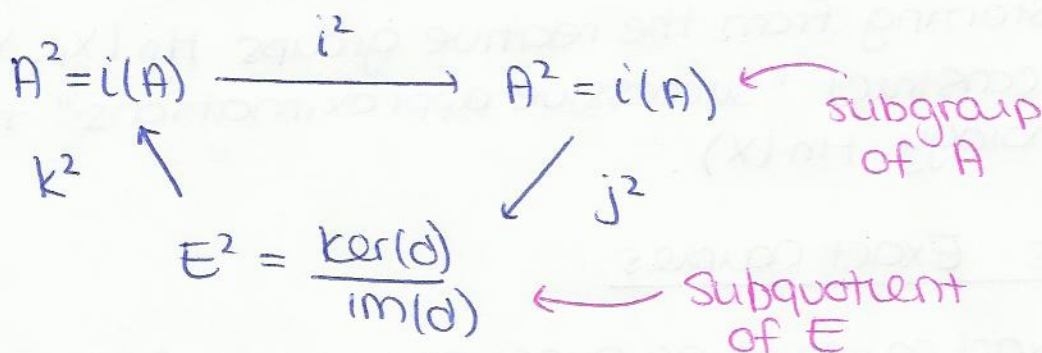
$k^2$  is the induced map on  $E^2$  by  $k$ .

$$j^2: A^2 \longrightarrow E^2$$

$$j^2: i(a) \longmapsto [j(a)]$$

It is straight-forward to check that these maps are well-defined.

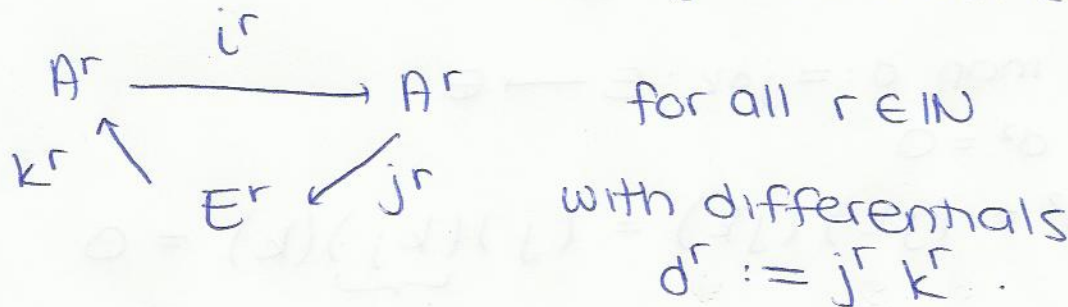
These definitions give the derived couple:



Claim The derived couple is an exact couple.

PF A diagram chase.

By iterating this construction, we get an exact couple



The sequence  $\{E^r, d^r\}$  is the data of the spectral sequence.

# The Spectral sequence of a filtered complex.

Given a finite filtration of topological spaces:

$$\emptyset \subseteq X_0 \subseteq X_1 \subseteq \dots \subseteq X_n = X$$

for each pair  $(X_p, X_{p-1})$  we have a long exact sequence:

$$\rightarrow H_n(X_{p-1}) \xrightarrow{i} H_n(X_p) \xrightarrow{j} H_n(X_p, X_{p-1}) \xrightarrow{k} H_{n-1}(X_{p-1}) \rightarrow$$

which we can condense:

$$\begin{array}{ccc}
 H_*(X_{p-1}) & \xrightarrow{i} & H_*(X_p) \\
 & \swarrow k & \searrow j \\
 & H_*(X_p, X_{p-1}) &
 \end{array}$$

which gives the following complex:

$$\begin{array}{ccccccc}
 0 \longrightarrow & H_*(X_0) & \xrightarrow{i_1} & H_*(X_1) & \xrightarrow{i_2} & H_*(X_2) & \xrightarrow{i_3} & H_*(X_3) \longrightarrow \\
 & \swarrow k_1 & & \swarrow j_1 & & \swarrow k_2 & & \swarrow j_2 & & \swarrow k_3 & & \swarrow j_3 \\
 & & H_*(X_1, X_0) & & H_*(X_2, X_1) & & H_*(X_3, X_2) & & & & & & 
 \end{array}$$

which is exact around each triangle.

If we condense this further:

$$A = \bigoplus_p H_*(X_p) \quad E = \bigoplus_p H_*(X_p, X_{p-1})$$

then the result is our exact couple

$$\begin{array}{ccc}
 A & \xrightarrow{i} & A \\
 & \swarrow k & \searrow j \\
 & E &
 \end{array}$$

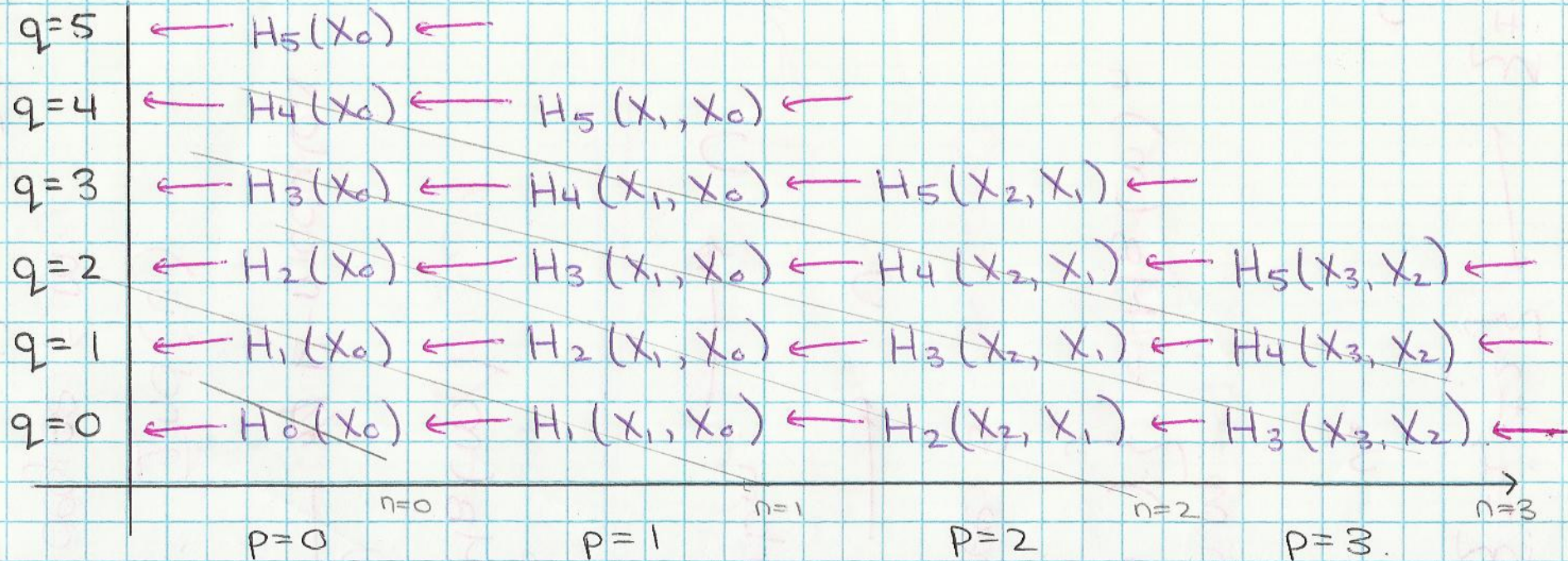
Degrees: Take  $q = n - p$ .

$i: (p-1, q) \rightarrow (p, q-1)$  degree  $(1, -1)$

$j: (p, q) \rightarrow (p, q)$  degree  $(0, 0)$

$k: (p, q) \rightarrow (p-1, q)$  degree  $(-1, 0)$ .

The  $E^1$  page : spectral sequence of a filtered complex  $X_p$



Our differentials have the form:

$$d^1 = jk = j_{p-1} \circ k_p$$

$$d^2 = j i^{-1} k = j_{p-2} \circ i_{p-1}^{-1} \circ k_p$$

$$\dots$$

$$d^r = j i^{-(r-1)} k = j_{p-r} \circ i_{p-r+1}^{-1} \circ k_p$$

degree  
(-r, r-1).

Note: Abuse of notation — the map  $i$  may not be invertible, but the claim is that these composites are well-defined.

Evolution of the sequence in an easy case:

Consider the filtration of spaces  $0 \subseteq X_0 \subseteq X_1 \subseteq X$

$$A_1: 0 \hookrightarrow H_*(X_0) \xrightarrow{i} H_*(X_1) \xrightarrow{i} H_*(X) \xrightarrow{\cong} H_*(X) \xrightarrow{\cong} H_*(X)$$

Composing the image of this map,  $i H_*(X_1)$ , with an isomorphism gives the inclusion of the image.

$$A_2: 0 \hookrightarrow \begin{matrix} i H_*(X_0) \\ \cap \\ H_*(X_1) \end{matrix} \rightarrow \begin{matrix} i H_*(X_1) \\ \cap \\ H_*(X) \end{matrix} \hookrightarrow H_*(X) \xrightarrow{\cong} H_*(X) \xrightarrow{\cong} H_*(X)$$

inclusion.

$$A_3: 0 \hookrightarrow \begin{matrix} i^2 H_*(X_0) \\ \cap \\ H_*(X) \end{matrix} \hookrightarrow \begin{matrix} i^2 H_*(X_1) \\ \cap \\ H_*(X) \end{matrix} \hookrightarrow H_*(X) \xrightarrow{\cong} H_*(X) \xrightarrow{\cong} H_*(X)$$

inclusion      inclusion

By  $A_3$ , all maps are injective.

Since the filtration is finite, the sequence eventually "runs into" isomorphisms, and stabilizes at a filtration of  $H_*(X)$  by the images of the absolute groups  $H_*(X_p)$ .

$$\begin{array}{ccccccc}
 A_3 & 0 & \hookrightarrow & i^2 H_*(X_0) & \xrightarrow{i^3} & i^2 H_*(X_1) & \xrightarrow{i^3} & H_*(X) & \xrightarrow{\cong} & H_*(X) \\
 & & & \uparrow k^3 & & \swarrow j^3 & & \uparrow k^3 & & \swarrow j^3 \\
 & & & E^3 & & E^3 & & E^3 & & E^3
 \end{array}$$

Since  $i^3$  is injective, by exactness,  $k^3 \equiv 0$  and  $j^3$  surjects

thus each  $E_p^3 \cong i^2 H_*(X_p) / i^2 H_*(X_{p-1})$  [by exactness at  $i^2 H_*(X_p)$ ]

In general, we find  $E_{p,q}^\infty \cong F_n^p / F_n^{p-1}$  ( $n=p+q$ )

for filtration  $0 \subseteq F_n^0 \subseteq \dots \subseteq F_n^n = H_n(X)$

with  $F_n^p = \text{Im} \left\{ H_n(X_p) \xrightarrow{i} H_n(X) \right\}$

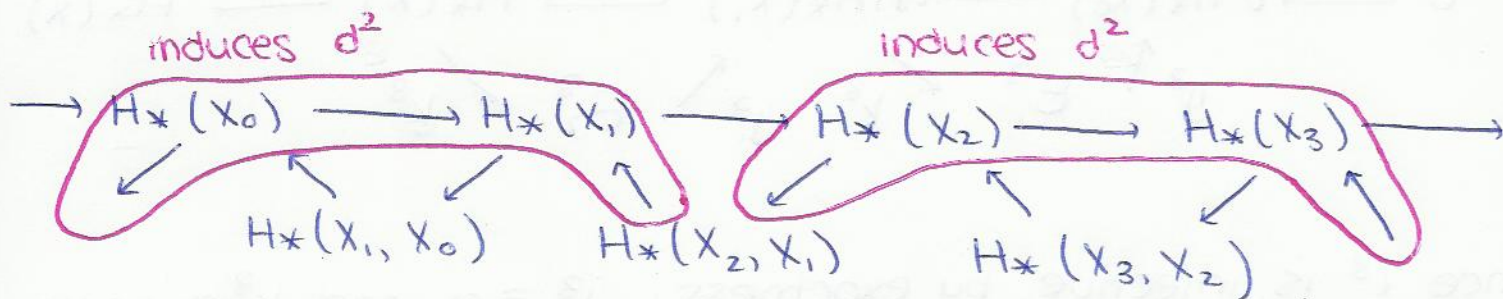
There is an analogous construction in cohomology with maps reversed,

in this case  $E_{p,q}^\infty \cong F_p^n / F_{p-1}^n$  ( $n=p+q$ )

for filtration  $0 \subseteq F_0^n \subseteq \dots \subseteq F_n^n = H^n(X)$

with  $F_p^n = \text{Ker} \left\{ H^n(X) \xrightarrow{i} H^n(X_p) \right\}$

What is happening at each step?



$$d^r = j \circ i^{-(r-1)} \circ k$$

Goal: Identify elements of  $H^*(X_p, X_{p-1})$  that lift (through  $j$ ) to  $H^*(X_p)$ , and survive (through  $i^{n-p}$ ) to  $H^*(X)$ .

Given  $x \in H^*(X_p, X_{p-1})$

- If  $x \notin \ker(d^r)$  for some  $r$ , then  $x \notin \ker(k) = \text{im}(j)$ , and  $x$  does not lift to  $H^*(X_p)$ .
- If  $x \in \text{im}(d^r)$  for some  $r$ ,  $x = j \circ i^{-(r-1)} \circ k(y)$ , then  $x$  lifts through  $j$  to  $i^{-(r-1)} \circ (k(y))$ , which maps through  $i^{r-1}$  to  $k(y) \in \text{im}(k) = \ker(i)$ . The lift of  $x$  does not represent an element of  $H^*(X)$ .
- Otherwise —  $x$  survives the spectral sequence to  $E^\infty$

### Exercise

Verify the following: If  $X$  is a CW-complex with  $p$ -skeleton  $X_p$

$$\text{then } E'_{p,q} = H_{p+q}(X_p, X_{p-1}) = \begin{cases} \bigoplus_{p\text{-cells}} \mathbb{Z} & , q=0 \\ 0 & \text{otherwise} \end{cases}$$

- $d^1$  is the usual differential from cellular homology
- the spectral sequence degenerates at  $E^2 = E^\infty$ ; all higher differentials are 0.

## Constructing the Leray-Serre spectral sequence.

(4)

The Leray-Serre spectral sequence arises as a special case of the spectral sequence of a filtered complex.

Given a fibration  $F \longrightarrow X \xrightarrow{\pi} B$

Suppose  $B$  is a CW-complex (or else pull back the fibration to CW approximation of  $B$ )

- Filter  $B$  by its  $p$ -skeleta  $B^p$
- Filter  $X$  by their inverse images  $\pi^{-1}(B^p) = X^p$

The Leray-Serre spectral sequence is the sequence associated to the filtration  $X^p$  of  $X$ .