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A Gentle Introduction to Spectral Sequence:

Constructing Spectral Sequences.

21 April 2011

Goal: Constructing the spectral sequence associated to a filtered space $\emptyset \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$

We will assume
- finite filtration
- spaces have homology groups of finite dimension/rank.

Intuition: Starting from the relative groups $H_n(X_p, X_{p-1})$, we will construct "successive approximations" to the homology $H_n(X)$.

Preliminaries: Exact Couples.

Defn An exact couple is an exact sequence of abelian groups of the form:

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

with i, j, k group homomorphisms.

Define map $d := j \circ k : E \longrightarrow E$.

Claim $d^2 = 0$

Pf $d^2 = (jk)(jk) = (j)\underbrace{(kj)}_{0}(k) = 0$

0 by exactness.

We will give a procedure for extending (E, d) to a spectral sequence.

Make the following definitions:

$$E^2 = \frac{\ker d}{\operatorname{im} d} \qquad A^2 = i(A)$$

$i^2 : A^2 \longrightarrow A^2$ $\qquad$ $i^2 = i\big|_{i(A)}$ $\qquad$ *NB the superscripts are indices, and not exponents.

$k^2 : E^2 \longrightarrow A^2$ $\qquad$ k^2 is the induced map on E^2 by k.

$j^2 : A^2 \longrightarrow E^2$

$$j^2 : i(a) \longmapsto [j(a)]$$

It is straight-forward to check that these maps are well-defined.

These definitions give the derived couple:

$$\begin{tikzcd} A^2 = i(A) \arrow[rr, "i^2"] & & A^2 = i(A) \arrow[dl, "j^2"] \\ & E^2 = \dfrac{\ker(d)}{\operatorname{im}(d)} \arrow[ul, "k^2"] & \end{tikzcd}$$

subgroup of A

subquotient of E

Claim The derived couple is an exact couple.

Pf A diagram chase.

By iterating this construction, we get an exact couple

$$\begin{tikzcd} A^r \arrow[rr, "i^r"] & & A^r \arrow[dl, "j^r"] \\ & E^r \arrow[ul, "k^r"] & \end{tikzcd}$$

for all $r \in \mathbb{N}$

with differentials $d^r := j^r k^r$.

The sequence $\{E^r, d^r\}$ is the data of the spectral sequence.

The spectral sequence of a filtered complex.

Given a finite filtration of topological spaces:

$$\emptyset \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$$

for each pair (X_p, X_{p-1}) we have a long exact sequence:

$$\to H_n(X_{p-1}) \xrightarrow{i} H_n(X_p) \xrightarrow{j} H_n(X_p, X_{p-1}) \xrightarrow{k} H_{n-1}(X_{p-1}) \to$$

which we can condense:

$$\begin{tikzcd} H_*(X_{p-1}) \arrow[rr, "i"] & & H_*(X_p) \arrow[dl, "j"] \\ & H_*(X_p, X_{p-1}) \arrow[ul, "k"] & \end{tikzcd}$$

which gives the following complex:

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & H_*(X_0) \arrow[rr, "i_1"] \arrow[dr, "j_1"] & & H_*(X_1) \arrow[rr, "i_2"] \arrow[dr, "j_2"] & & H_*(X_2) \arrow[rr, "i_3"] \arrow[dr, "j_3"] & & H_*(X_3) \arrow[r] & {} \\ & & H_*(X_1, X_0) \arrow[ul, "k_1"] & & H_*(X_2, X_1) \arrow[ul, "k_2"] & & H_*(X_3, X_2) \arrow[ul, "k_3"] & & \end{tikzcd}$$

which is exact around each triangle.

If we condense this further:

$$A = \bigoplus_p H_*(X_p) \qquad E = \bigoplus_p H_*(X_p, X_{p-1})$$

then the result is our exact couple

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Degrees: Take $q = n - p$.

$i : (p-1, q) \longrightarrow (p, q-1)$ degree $(1, -1)$

$j : (p, q) \longrightarrow (p, q)$ degree $(0, 0)$

$k : (p, q) \longrightarrow (p-1, q)$ degree $(-1, 0)$.

The E^1 page: spectral sequence of a filtered complex X_p

$$
\begin{array}{l|ccccccccc}
q=5 & \longleftarrow & H_5(X_0) & \longleftarrow & & & & & & \\
q=4 & \longleftarrow & H_4(X_0) & \longleftarrow & H_5(X_1,X_0) & \longleftarrow & & & & \\
q=3 & \longleftarrow & H_3(X_0) & \longleftarrow & H_4(X_1,X_0) & \longleftarrow & H_5(X_2,X_1) & \longleftarrow & & \\
q=2 & \longleftarrow & H_2(X_0) & \longleftarrow & H_3(X_1,X_0) & \longleftarrow & H_4(X_2,X_1) & \longleftarrow & H_5(X_3,X_2) & \longleftarrow \\
q=1 & \longleftarrow & H_1(X_0) & \longleftarrow & H_2(X_1,X_0) & \longleftarrow & H_3(X_2,X_1) & \longleftarrow & H_4(X_3,X_2) & \longleftarrow \\
q=0 & \longleftarrow & H_0(X_0) & \longleftarrow & H_1(X_1,X_0) & \longleftarrow & H_2(X_2,X_1) & \longleftarrow & H_3(X_3,X_2) & \longleftarrow \\
\hline
 & & n=0 & & n=1 & & n=2 & & n=3 & \\
 & & p=0 & & p=1 & & p=2 & & p=3 &
\end{array}
$$

Our differentials have the form:

$$d^1 = jk = j_{p-1} \circ k_p$$

$$d^2 = j i^{-1} k = j_{p-2} \circ i^{-1}_{p-1} \circ k_p$$

...

$$d^r = j\, i^{-(r-1)}\, k = j_{p-r} \circ i^{-r+1} \circ k_p.$$

degree $(-r, r-1)$.

Note: Abuse of notation — the map i may not be invertible, but the claim is that these composites are well-defined.

Evolution of the sequence in an easy case:

Consider the filtration of spaces $0 \subseteq X_0 \subseteq X_1 \subseteq X$

$$A_1: \quad 0 \hookrightarrow H_*(X_0) \xrightarrow{i} H_*(X_1) \xrightarrow{i} H_*(X) \xrightarrow{\cong} H_*(X) \xrightarrow{\cong}$$

composing the image of this map, $i H_*(X_1)$, with an isomorphism gives the inclusion of the image.

$$A_2: \quad 0 \hookrightarrow \underset{H_*(X_1)}{\overset{\cap}{i H_*(X_0)}} \longrightarrow \underset{H_*(X)}{\overset{\cap}{i H_*(X_1)}} \underset{\text{inclusion}}{\hookrightarrow} H_*(X) \xrightarrow{\cong} H_*(X) \xrightarrow{\cong}$$

$$A_3: \quad 0 \hookrightarrow \underset{H_*(X)}{\overset{\cap}{i^2 H_*(X_0)}} \underset{\text{inclusion}}{\hookrightarrow} \underset{H_*(X)}{\overset{\cap}{i^2 H_*(X_1)}} \underset{\text{inclusion}}{\hookrightarrow} H_*(X) \xrightarrow{\cong} H_*(X) \xrightarrow{\cong}$$

By A_3, all maps are injective.

Since the filtration is finite, the sequence eventually "runs into" isomorphisms, and stabilizes at a filtration of $H_*(X)$ by the images of the absolute groups $H_*(X_p)$.

$$A_3 \quad \begin{tikzcd} 0 \arrow[r, hook] & i^2 H_*(X_0) \arrow[rr, hook, "i^3"] & & i^2 H_*(X_1) \arrow[rr, hook, "i^3"] \arrow[dl, "j^3"] & & H_*(X) \arrow[rr, "\cong"] \arrow[dl, "j^3"] & & H_*(X) \\ & & E^3 \arrow[ul, "k^3"] & & E^3 \arrow[ul, "k^3"] & & & \end{tikzcd}$$

Since i^3 is injective, by exactness, $k^3 \equiv 0$ and j^3 surjects

thus each $E^3_p \cong i^2 H_*(X_p) / i^2 H_*(X_{p-1})$. [by exactness at $i^2 H_*(X_p)$]

In general, we find $E^{\infty}_{p,q} \cong F^p_n / F^{p-1}_n \quad (n=p+q)$

for filtration $0 \subseteq F^0_n \subseteq \cdots \subseteq F^n_n = H_n(X)$

with $F^p_n = \mathrm{Im}\left\{ H_n(X_p) \xrightarrow{i} H_n(X) \right\}$

There is an analogous construction in cohomology with maps reversed,

in this case $E_{\infty}^{p,q} \cong F^n_p / F^n_{p-1} \quad (n=p+q)$

for filtration

$$0 \subseteq F^n_0 \subseteq \cdots \subseteq F^n_n = H^n(X)$$

with $F^n_p = \mathrm{Ker}\left\{ H^n(X) \xrightarrow{i} H^n(X_p) \right\}$

What is happening at each step?

induces d^2 induces d^2

$$\begin{tikzcd}
{} \arrow[r] & H_*(X_0) \arrow[rr] & & H_*(X_1) \arrow[rr] \arrow[dl] & & H_*(X_2) \arrow[rr] \arrow[dl] & & H_*(X_3) \arrow[r] \arrow[dl] & {} \\
& & H_*(X_1, X_0) \arrow[ul] & & H_*(X_2, X_1) \arrow[ul] & & H_*(X_3, X_2) \arrow[ul] & &
\end{tikzcd}$$

$$d^r = j \circ i^{-(r-1)} \circ k$$

Goal: Identify elements of $H_*(X_p, X_{p-1})$ that lift (through j) to $H_*(X_p)$, and survive (through i^{n-p}) to $H_*(X)$.

Given $x \in H_*(X_p, X_{p-1})$

- If $x \notin \ker(d^r)$ for some r,
  then $x \notin \ker(k) = \operatorname{im}(j)$, and x does not lift to $H_*(X_p)$.

- If $x \in \operatorname{im}(d^r)$ for some r,
  $x = j \circ i^{-(r-1)} \circ k(y)$,
  then x lifts through j to $i^{-(r-1)} \circ (k(y))$,
  which maps through i^{r-1} to $k(y) \in \operatorname{im}(k) = \ker(i)$.
  The lift of x does not represent an element of $H_*(X)$.

- Otherwise — x survives the spectral sequence to E^∞.

Exercise

Verify the following: If X is a CW-complex with p-skeleta X_p

then $E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) = \begin{cases} \bigoplus_{p\text{-cells}} \mathbb{Z} & , q = 0 \\ 0 & \text{otherwise} \end{cases}$

- d^1 is the usual differential from cellular homology
- the spectral sequence degenerates at $E^2 = E^\infty$,
  all higher differentials are 0.

Constructing the Leray-Serre spectral sequence.

The Leray-Serre spectral sequence arises as a special case of the spectral sequence of a filtered complex.

Given a fibration $F \longrightarrow X \xrightarrow{\pi} B$

Suppose B is a CW-complex (or else pull back the fibration to CW approximation of B)

- Filter B by its p-skeleta B^p
- Filter X by their inverse images $\pi^{-1}(B^p) = X^p$

The Leray-Serre spectral sequence is the sequence associated to the filtration X^p of X.