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The top-degree cohomology of the special linear group of a number field

Talk #1

F - number field (finite field extension of $\mathbb{Q}$)
eg. $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{5})$, ...
R - ring of integers of F (ie, all solutions over F to monic poly's with coefficients in $\mathbb{Z}$).
eg. $\mathbb{Z} \subseteq \mathbb{Q}$
$\mathbb{Z}[i] \subseteq \mathbb{Q}(i)$
$\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right] \subseteq \mathbb{Q}(\sqrt{5})$

Recall from Mikala's talk:

Thm (Borel-Serre)

$$\mathrm{vcd}(SL_n R) = r\binom{n+1}{2} + cn^2 - n - r - c + 1$$

r = # embeddings $F \hookrightarrow \mathbb{R}$
c = # complex-conjugate pairs of embeddings $F \hookrightarrow \mathbb{C}$ that do not factor thru $\mathbb{R}$

quadratic in n,
eg vcd of $SL_n\mathbb{Z}$ is $\binom{n}{2}$.

SL_nR is a virtual duality gp. with dualizing module the Steinberg representation $St_n(F)$.
[defined shortly]

This means:

$$H^{\mathrm{vcd}-i}(SL_nR; V) \cong H_i(SL_nR; V \otimes_{\mathbb{Z}} St_n(F))$$

for all $\mathbb{Q}[SL_nR]$-modules V.

In particular

$$H^{\mathrm{vcd}-i}(SL_nR; \mathbb{Q}) \cong H_i(SL_nR; \mathbb{Q} \otimes_{\mathbb{Z}} St_n(F))$$

To compute these groups: [from the defn of group homology].

- Take resolution of $\mathbb{Q} \otimes_{\mathbb{Z}} St_n(F)$ by flat $\mathbb{Q}[SL_nR]$-modules
- Take SL_nR - coinvariants
- Take homology

So the key to computing the rational cohomology of SL_nR near its vcd is to compute a "nice" flat resolution of the Steinberg module, where "nice" means it is tractable to compute the coinvariants of its terms

Defn of the Steinberg module

Tits bldg $T_n(F)$ - simplicial complex
vertices $\longleftrightarrow$ subspaces $0 \subsetneq V \subsetneq F^n$
p-simplices $\longleftrightarrow$ flags $0 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_p \subsetneq F^n$

Natural SL_nR-action.

Ie, $T_n(F)$ is the geometric realization of the poset of proper nonzero subspaces of F^n under inclusion.

Thm (Solomon-Tits) $T_n(F) \simeq \bigvee S^{n-2}$

The Tits bldg is a wedge of infinitely many spheres of dim $(n-2)$.

[proof in exercises].

$\rightsquigarrow$ Reduced homology only nonzero in degree $(n-2)$.

Defn $St_n(F) = \tilde{H}_{n-2}(T_n(F))$

$\rightsquigarrow$ key to computing "nice" resolution of Steinberg module is to understand the topology of the Tits buildings.

In Exercises: construct Sharbly resolution

Question: What is the largest degree $SL_n R$ has nonvanishing rational cohomology?

Known results:

$H^{vcd}(SL_n R; \mathbb{Q}) = 0$ • R Euclidean $(n \geq 2)$ [Lee-Szczarba]

$H^{vcd}(SL_n R; \mathbb{Q}) \neq 0$ • R not a PID [Church-Farb-Putman]

• n even, $F = \mathbb{Q}(\sqrt{d})$ for $d = -43, -67, -163$ [Miller-Patzt-W-Yasaki]

unknown for n odd.

top degree.

↑ Assuming the generalized Riemann hypothesis, the only number ring not on this list is $d = -19$.

$H^{vcd-1}(SL_n R; \mathbb{Q}) = 0$ $(n \geq 3)$. • $R = \mathbb{Z}$ [Church-Putman]

• R = Gaussian or Eisenstein integers $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-3})$ [Kupers-Miller-Patzt-W].

codim 1

Today's Goal: Lee-Szczarba

Assume R Euclidean (eg, $\mathbb{Z}, \mathbb{Z}[i], \mathbb{Z}[\sqrt{-2}], \ldots$)

$$H^{vcd}(SL_n R; \mathbb{Q}) \cong H_0(SL_n R; St_n(F) \otimes_{\mathbb{Z}} \mathbb{Q})$$
$$\cong (St_n(F) \otimes_{\mathbb{Z}} \mathbb{Q})_{SL_n R}$$

• Strategy: find generators of $St_n(F)$ that vanish in the $SL_n R$-coinvariants.

$$\boxed{\begin{array}{c}\text{Coinvariants:} \\ G \curvearrowright M \\ M_G := M/\langle m - gm \rangle \\ \text{largest quotient with trivial } G\text{-action.}\end{array}}$$

Case n=2

$T_2(F) = \{\text{lines in } F^2\}$ discrete set

$\tilde{H}_0(T_2(F)) = \langle L_1 - L_0 \rangle$ $L_i \subseteq F^2$ line

Eg $R = \mathbb{Z}$ $x = \mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} - \mathbb{Q}\begin{bmatrix}0\\1\end{bmatrix}$ $g = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \in SL_2\mathbb{Z}$

Then $g \cdot x = -x$ g interchanges the two lines

so x vanishes in $SL_2\mathbb{Z}$-coinvariants.

Eg $R = \mathbb{Z}$, $y = \mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} - \mathbb{Q}\begin{bmatrix}1\\2\end{bmatrix}$

Problem: there is no $g \in SL_2\mathbb{Z}$ that interchanges these lines (Exercise!)

Why? $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}$ are not a basis for $\mathbb{Z}^2$.

↑ I chose primitive vectors in each line, but these vectors are cols of a matrix of det 2.

ie, if I intersect each line with $\mathbb{Z}^2$, the direct sum is an index-2 subgroup of $\mathbb{Z}^2$.

$\left(\mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} \cap \mathbb{Z}^2\right) \oplus \left(\mathbb{Q}\begin{bmatrix}1\\2\end{bmatrix} \cap \mathbb{Z}^2\right)$ is index 2 in $\mathbb{Z}^2$.

Key claim: When R is Euclidean,

$St_n(F)$ is generated by $\{ L_1 - L_2 \mid (L_1 \cap R^2) \oplus (L_1 \cap R^2) = R^2 \}$

ie, $L_1 = Fv_1$, $L_2 = Fv_2$ for some basis v_1, v_2 of R^2.

These generators are called <u>integral apartment classes</u>.

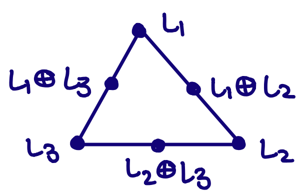
<u>Def^n</u> Given a frame for F^n, $F^n = L_1 \oplus L_2 \oplus \cdots \oplus L_n$

A <u>frame</u> is a direct sum decomposition into lines.

Let $S(L_1, L_2, \ldots, L_n) \subseteq T_n(F)$ be the full subcomplex on vertices indexed by direct sums of L_i's.

$S(L_1, \ldots, L_n)$ is called an <u>apartment</u>.

Exercises: $S(L_1, \ldots, L_n) \simeq S^{n-2}$.



Can identify $S(L_1, \ldots, L_n)$ with barycentric subdivision of boundary of $(n-1)$ simplex.

<u>Thm</u> (Solomon-Tits) $\tilde{H}_{n-2}(T_n(F))$ is gen by apartment classes.

<u>Thm</u> (Ash-Rudolph) When R is Euclidean, [not necessarily a number ring]

$\tilde{H}_{n-2}(T_n(F))$ is gen. by <u>integral</u> apartment classes

ie, $[S(L_1, \ldots, L_n)]$ where $(L_1 \cap R^n) \oplus \cdots \oplus (L_n \cap R^n) = R^n$ ← Comes from an R-basis for R^n.

<u>Proof of Lee-Szczarba</u> (assuming Ash-Rudolph)

- suffices to show integral apartment classes vanish in SL_nR-coinvariants.
- Let $L_1 \oplus L_2 \oplus \cdots \oplus L_n = F^n$ be an integral frame, ie, $L_i = Fv_i$ for some basis $\{v_1, \ldots, v_n\}$ of R^n.

Let $g = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & Id \end{array}\right] \in SL_nR$ written wrt basis $v_1, \ldots, v_n$

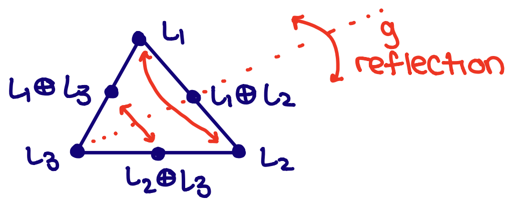
so g: $L_1 \longmapsto L_2$

$L_2 \longmapsto L_1$

$L_i \longmapsto L_i \quad \forall i \geq 3$

Then g stabilizes the corresponding apartment and acts by orientation-reversing automorphism.

$\rightsquigarrow \quad g \cdot [S(L_1, \dots, L_n)] = -[S(L_1, \dots L_n)]$ $\quad$ g negates the apartment class.



$\rightsquigarrow$ integral apartment classes vanish in coinvariants $(St_n(F) \otimes_{\mathbb{Z}} \mathbb{Q})_{SL_n R}$

$\rightsquigarrow H^{vcd}(SL_n R; \mathbb{Q}) = 0$ for R Euclidean.