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A generating set for the Steinberg module of a Euclidean ring

Talk #2 Review

R = ring of integers in number field F.

virtual Bieri-Eckmann duality: $H^{vcd-i}(SL_n R; \mathbb{Q}) \cong H_i(SL_n R; \mathbb{Q} \otimes_{\mathbb{Z}} St_n(F))$

Dualizing module: $St_n(F) = \tilde{H}_{n-2}(T_n(F))$

$T_n(F)$ = Tits building

vertices $\longleftrightarrow$ proper nonzero subspaces of F^n
simplices $\longleftrightarrow$ flags

Thm (Solomon-Tits)
- $T_n(F) \simeq \bigvee S^{n-2}$
- $\tilde{H}_{n-2}(T_n(F))$ is generated by apartment classes.

apartments $\longleftrightarrow$ frames $F = L_1 \oplus L_2 \oplus \cdots \oplus L_n$

$S(L_1, \ldots, L_n)$ = full subcomplex of $T_n(F)$ on vertices corresponding to direct sums of lines L_i.

Thm (Ash-Rudolph) R Euclidean
Then $\tilde{H}_{n-2}(T_n(F))$ is gen by integral apartment classes,
ie, $[S(L_1, \ldots, L_n)]$ where $(L_1 \cap R^n) \oplus \cdots \oplus (L_n \cap R^n) = R^n$
The frame arises from a basis for R^n.

Thm (Lee-Szczarba) R Euclidean. $H^{vcd}(SL_n R; \mathbb{Q}) = 0$

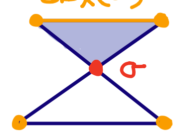
Last Time: proof that Ash-Rudolph $\Rightarrow$ Lee-Szczarba.

Today: Simplified proof of Ash-Rudolph due to Church-Farb-Putman.

Simplicial methods.

Defn X simplicial cmplx, σ in X simplex, $\sigma = [s_0, \dots, s_p]$

The link of σ is $\mathrm{Link}_X(\sigma) =$ subcomplex on simplices $\{[t_0, \dots, t_q] \mid [s_0, \dots, s_p, t_0, \dots, t_q]$ is a simplex in $X\}$.



Link is all faces opposite σ in simplices containing σ as a subsimplex.

Defn X d-dim'l simplicial complex

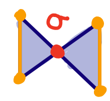
X is Cohen-MacCaulay (CM) if

- X is $(d-1)$-connected
- $\mathrm{Link}_X(\sigma)$ is $(d-2-\dim(\sigma))$-connected $\forall$ simplices σ

Eg the standard simplicial structure on ball, sphere

 CM  CM.

Eg. Not CM even though contractible



$\mathrm{Link}_X(\sigma)$ is disconnected.

[The original (inequivalent) defn of CM was only a condition on homology of links (not π_1). That version was a homeo invariant, this defn - due to Quillen - is not.]

Complex of partial bases

Fix R - Euclidean ring. F - field of fractions

Defn $v_0, \dots, v_p \in R^n$ is a partial basis if it is a subset of a basis (possibly equal to a basis)

Eg (Exercise) $\{v_0\}$ is a partial basis $\Leftrightarrow$ v_0 primitive (its components generate R).

Eg (Exercise) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$ are not partial bases of $\mathbb{Z}^2$.

Defn $B_n(R)$ - complex of partial bases $(n-1)$-dim'l simplicial complex

vertices — primitive $v_0 \in R^n$

simplices — partial bases.

Thm (Maazen 1979) R Euclidean. The barycentric subdivision of $B_n(R)$ is CM.

$B_n(R)$ is as highly connected as a wedge of $(n-1)$-dim'l spheres.

Goal: Relate Tits building to $B_n(R)$ to get nice generating set for its homology.

Key: Quillen's lemma.

Quillen's lemma

Notation X poset, $|X|$ geometric realization.

Lemma (Quillen 1978) $f: X \to Y$ strictly increasing map of posets

- $|Y|$ CM, dim d
- $\forall y \in Y$, fibre $f_y = \{x \in X \mid f(x) \leq y\}$ has $|f_y|$ CM

Then $|X|$ is CM, and $f_*: \tilde{H}_d(|X|) \to \tilde{H}_d(|Y|)$ surjects.

Proof of Ash-Rudolph

Y = poset of proper nonzero summands of F^n
(so $|Y| = T_n(F)$, dim $(n-2)$)

X = proper partial bases of R^n under inclusion
(so $|X|$ = barycentric subdivision of $(n-2)$-skeleton of $B_n(R)$)

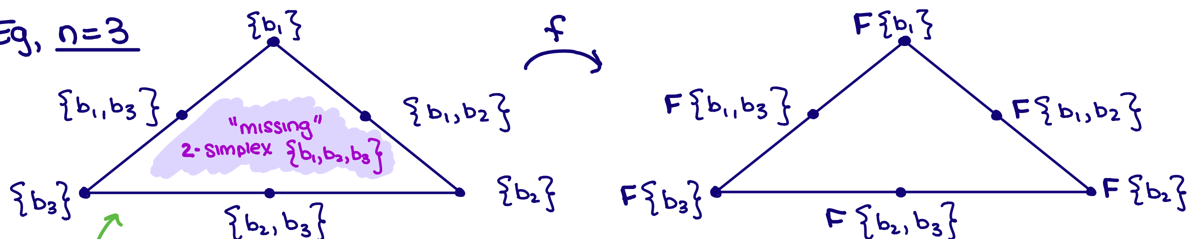
$$f : X \longrightarrow Y$$
$$\{v_0, \dots, v_p\} \longmapsto \mathrm{span}_F\{v_0, \dots, v_p\}$$

Check hypotheses of lemma:

- map f is strictly increasing [strict containment of partial bases $\rightsquigarrow$ strict containment of subspaces]
- $|Y|$ CM [Solomon-Tits].
- $\forall\, V \subseteq F^n$, $f_V = \{$partial bases contained in $V\}$ = barycentric subdivision of partial basis complex $B(V)$.
  $|f_V|$ is CM by Maazen.

$\rightsquigarrow$ Lemma $\Rightarrow$ $\tilde{H}_{n-2}(|X|) \longrightarrow \tilde{H}_{n-2}(T_n(F))$ surjects.

Eg, $n=3$



integral apartment.

Since $B_n(R)$ is $(n-2)$-connected, we might expect the deg $(n-2)$ homology of its $(n-2)$-skeleton to come from the boundaries of the "missing" $(n-1)$ simplices. We will verify this.

Then we're done – the $(n-1)$ simplices correspond to bases of R^n, so their images are integral apartments.
By Quillen lemma, these integral apartment classes generate $\tilde{H}_{n-2}(T_n(F))$.

From LES of a pair: $\tilde{H}_{n-1}(B_n(R), |X|) \rightarrow \tilde{H}_{n-2}(|X|) \rightarrow \cancel{\tilde{H}_{n-2}(B_n(R))}$

0 by Maazen

surjects by exactness.

$\boxed{\text{connecting homomorphism: takes an } (n-1)\text{-simplex of } B_n(R) \text{ to its boundary in } |X|}$

So we have $\tilde{H}_{n-1}(B_n(R), |X|) \twoheadrightarrow \tilde{H}_{n-2}(|X|) \twoheadrightarrow H_{n-2}(T_n(F))$

surjects by Quillen Lemma

$\|$

group of $(n-1)$-chains on $B_n(R)$

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Free abelian gp on bases for R^n

since $|X|$ is the codimension-1 skeleton of $B_n(R)$.

Conclusion: $\tilde{H}_{n-2}(T_n(F))$ is generated by integral apartment classes.

Time permitting: <u>Proof of Maazen, n=2.</u>

<u>Exercise</u>: It suffices to show $B_2(R)$ is a connected graph.

Let $\begin{bmatrix} a \\ b \end{bmatrix} \in R$ be primitive. <u>Goal</u>: build path to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

<u>Claim</u>: We can use the Euclidean algorithm to build an edge
$\begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix}$ with $|d| < |b|$, $|\cdot|$ the Euclidean norm.

<u>Pf of claim</u>: Choose any basis $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c' \\ d' \end{bmatrix}$. $\exists\, q$ s.t. $|d' - qb| < |b|$
Take $\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} c' \\ d' \end{bmatrix} - q \begin{bmatrix} a \\ b \end{bmatrix}$. Check: this is still a basis.

Iterate until $d=0$.