Jenny Wilson Farb & Friends Student Seminar 30 Jan 2014

Braid groups, representation stability, and polynomials over $\mathbb{F}_q$

The goal of this talk is to describe a (surprising) relationship between the structure of the cohomology of the braid group B_n with local coefficients in S_n-representations, and certain statistics for polynomials over finite fields.

Specifically, let X denote the space of monic square-free deg n polynomials with coefficients in $\mathbb{F}_q$. Let P be polynomial in the number of irreducible factors of $f \in X$ of given lengths. There is a way to associate an S_n-rep V over $\mathbb{C}$ to P, and

$$\sum_{f \in X} P(f) = \sum_{i \geq 0} (-1)^i q^{n-i} \dim_{\mathbb{C}} H^i(B_n; V)$$

where $H^{n-i}(B_n; V)$ is the cohomology of B_n with twisted coefficients in V. (All this terminology will be defined).

This talk is based on a paper by Church-Ellenberg-Farb, "Representation Stability in cohomology and asymptotics for families of varieties over finite fields"

ArXiv: 1309.6038

In this talk, we will

① Recall the definition of configuration space and its relationship to the (pure) braid group and moduli spaces of polynomials.

② Review the definitions of (co)homology with local coefficients, and the transfer maps

③ Overview Étale cohomology and the Grothendieck-Lefschetz fixed point theorem

④ See implications of rep stability to polynomial statistics

Some Motivating Formulas:

Let X = space of monic deg n polynomials with coefficients in $\mathbb{F}_q$ (q prime)

B_n = Artin's braid group

Statistic on X	Count	Cohomology of B_n with local coeffs.	Ranks of ith cohomology groups
$\# \lvert X \rvert$	$q^n - q^{n-1}$	$H^*(B_n; \mathbb{C})$	$\beta_0 = \beta_1 = 1$ $\beta_i = 0, \ i > 1$
$\sum_{f \in X} \#$ linear factors of f	$q^n - 2q^{n-1} + 2q^{n-2} \cdots \pm 2q^2 \mp q$	$H^*(B_n; \mathbb{C}^n)$ $\mathbb{C}^n = V_{\square\square\square\square\square} \oplus V_{\begin{smallmatrix}\square\square\square\square\\ \square\end{smallmatrix}}$	$\beta_0 = \beta_{n-1} = 1$ $\beta_i = 2, \ i = 1, \ldots, n-2$ $\beta_i = 0, \ i \geq n$
$\sum_{f \in X} P(f)$ P poly in $\#$ deg k irreducible factors of f, $k = 1, \ldots, N$ for some N.	$\sum_{i \geq 0} (-1)^i \beta_i q^{n-i}$	$H^*(B_n; V_P)$ $\uparrow$ Cohomology of B_n with coeffs in S_n-rep V_P whose character is (somehow) determined by P.	β_i

Today's Main objects: Conf_n and B_n

Choose a field $\mathbb{F}$

Let $\mathrm{PConf}_n(\mathbb{F}) = \{(x_1, \dots, x_n) \in \mathbb{F}^n \mid x_i \neq x_j \text{ for } i \neq j\}$ denote the pure (ordered) configuration space of distinct points in $\mathbb{F}$.

For $\mathbb{F} = \mathbb{C}$, we topologize $\mathrm{PConf}_n(\mathbb{C})$ as a subspace of $\mathbb{C}^n$.

$\mathrm{PConf}_n(\mathbb{C})$ is a $K(\pi, 1)$-space for its fundamental group, the pure braid group P_n.

The symmetric group $S_n \curvearrowright \mathrm{PConf}_n(\mathbb{F})$ by permuting the n-tuples.

The quotient $\mathrm{Conf}_n(\mathbb{F}) := \mathrm{PConf}_n(\mathbb{F}) / S_n$ is the (unordered) configuration space of sets of n elements in $\mathbb{F}$.

$\mathrm{Conf}_n(\mathbb{C})$ is a $K(\pi, 1)$ for Artin's braid group B_n.

We thus have a normal covering map:

$$\begin{array}{ll} \mathrm{PConf}_n(\mathbb{C}) & \pi_1 = P_n \\ \Big\downarrow & \circlearrowleft \; S_n = B_n / P_n \quad \text{Deck group} \\ \mathrm{Conf}_n(\mathbb{C}) = \mathrm{PConf}_n(\mathbb{C}) / S_n & \pi_1 = B_n \end{array}$$

For a field $\mathbb{F}$, we can identify

$$\begin{array}{ccc} \mathrm{Conf}_n(\mathbb{F}) & \xleftrightarrow{\;=\;} & \text{monic deg-}n\text{ square-free polynomials with roots in } \mathbb{F} \\ \{x_1, \dots, x_n\} & \longleftrightarrow & f(t) = (t - x_1) \cdots (t - x_n) \end{array}$$

This space of polynomials is an algebraic variety, the complement of the vanishing set of the discriminant in coeffs of f.

Frobenius morphism.

Given a variety defined by polynomials over $\mathbb{F}_q$, (q prime) the Frobenius map acts on its $\overline{\mathbb{F}_q}$ points.

For $x \in \overline{\mathbb{F}_q}$, $\quad \mathrm{Frob}_q : x \longmapsto x^q$.

Frobenius fixes precisely the $\mathbb{F}_q$ points

(since $\mathrm{Frob}_q(x) = x \iff x$ is a root of $t^q - t$).

More generally, the mth iterate Frob_q^m fixes the points in $\mathbb{F}_{q^m}$. These iterates generate the Galois groups of field extensions of $\mathbb{F}_{q^m}$.

In our case, $\mathrm{Frob}_q \curvearrowright \mathrm{Conf}_n(\overline{\mathbb{F}_q})$.

A polynomial $f \in \mathrm{Conf}_n(\overline{\mathbb{F}_q})$ is fixed by Frob_q exactly when its coefficients lie in $\mathbb{F}_q$.

In this case, Frob_q permutes the roots of $f(t)$.

The orbits of roots under Frob_q correspond to the irreducible factors (over $\mathbb{F}_q$) of f.

Thus, for each $f \in \mathrm{Fix}(\mathrm{Frob}_q)$

Frob_q determines a permutation $\sigma_f \in S_n$ on the roots of f.

(The roots are unordered, so σ_f is defined up to conjugacy)

More on this soon.... first, a review of local coefficients.

(co) homology with local (twisted) coefficients.

Let X be a path-connected topological space, with $\pi_1 = \pi_1(X)$.

$$\begin{array}{ll} \tilde{X} & \circlearrowleft \pi_1 \\ \downarrow & \\ X & \end{array}$$

π_1 acts on the universal cover $\tilde{X}$ of X by Deck transformations.

This induces an action $\pi_1 \curvearrowright C_n(\tilde{X})$ on the singular chains on $\tilde{X}$.
This action commutes with the bdry maps.

Let M be any left $\mathbb{Z}[\pi_1]$-module.

Define $C_n(X;M) := C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1]} M$

$C^n(X;M) := \mathrm{Hom}_{\mathbb{Z}[\pi_1]}(C_n(\tilde{X}); M)$

The boundary maps ∂ on $C_n(\tilde{X})$ induce maps on $C_n(X;M)$ and $C^n(X;M)$ making these (co)-chain complexes.

<u>Def^n</u> $H_\bullet(X;M) := H_\bullet(C_\bullet(X;M))$

$H^\bullet(X;M) := H_\bullet(C^\bullet(X;M))$

The homology of these complexes is what we define to be the (co)-homology groups of X with <u>twisted</u> (or <u>local</u>) coefficients in M.

<u>Remark</u> In order to form the tensor product $C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1]} M$, we need to turn $C_n(\tilde{X})$ from a left to a right $\mathbb{Z}[\pi_1]$-module.

To do this we define: $g \cdot \sigma := \sigma \cdot g^{-1}$

Thus tensoring over $\mathbb{Z}[\pi_1]$ is tantamount to tensoring over $\mathbb{Z}$ and modding out by the diagonal action of π_1:

$$C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1]} M = \left(\langle \sigma \otimes m \rangle \Big/ \sigma \otimes m = \sigma g^{-1} \otimes g m \right)$$

<u>First examples</u> of (co)homology with twisted coefficients:

<u>Eg</u> $M = \mathbb{Z}$ trivial $\mathbb{Z}[\pi_1]$-module.

Then $C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1]} \mathbb{Z} \cong C_n(\tilde{X}) \Big/ \text{action of } \pi_1$

$\cong C_n(X)$

since we can identify chains in $C_n(X)$ with π_1-orbits of chains in $C_n(\tilde{X})$

so $H_\bullet(X; \mathbb{Z})$ is the usual homology of X with integer coefficients.

<u>Exercise</u> $H^\bullet(X; \mathbb{Z})$ is usual cohomology of X.

<u>Eg</u> $M = \mathbb{Z}[\pi_1]$ group ring.

Then $C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1]} \mathbb{Z}[\pi_1] \cong C_n(\tilde{X})$

and $H_\bullet(X; \mathbb{Z}[\pi_1]) \cong H_\bullet(\tilde{X}; \mathbb{Z})$.

<u>Eg</u> $M = \mathbb{Z}[\pi_1/\pi_1']$ for $\pi_1' \trianglelefteq \pi_1$

$$\begin{array}{ll} \tilde{X} & \\ \downarrow & \\ X' & \pi_1' \\ \downarrow & \\ X & \pi_1 \end{array}$$

Then $C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1]} \mathbb{Z}[\pi_1/\pi_1']$

$\cong C_n(\tilde{X})/\pi_1' \cong C_n(X')$

and $H_\bullet(X, M) \cong H_\bullet(X'; M)$

normal cover assoc. to π_1'

More generally, if M is a $\mathbb{Z}[\pi_1]$-module such that some normal subgp $\pi_1' \trianglelefteq \pi_1$ acts trivially on M (ie, the action $\pi_1 \curvearrowright M$ factors through π_1/π_1')

$$\begin{array}{ccl} & \tilde{X} & \circlearrowleft \text{Deck gp } \pi_1' \\ & \downarrow & \\ \pi_1' & X' & \circlearrowleft \text{Deck gp } \pi_1/\pi_1' \\ & \downarrow & \\ \pi_1 & X & \end{array}$$

normal covering spaces

Then
$$C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1]} M$$
$$\cong C_n(X') \otimes_{\mathbb{Z}[\pi_1/\pi_1']} M$$

Upshot: To understand $H_\bullet(X; M)$ for a $\mathbb{Z}[\pi_1/\pi_1']$-module M, it suffices to understand the action of π_1/π_1' on the cover X'.

In our case:

$$\begin{array}{cl} \mathrm{PConf}_n(\mathbb{C}) & \circlearrowleft S_n = B_n/P_n \\ \downarrow & \\ \mathrm{Conf}_n(\mathbb{C}) & \end{array}$$

To understand $H_\bullet(\mathrm{Conf}_n(\mathbb{C}); V)$ and $H^\bullet(\mathrm{Conf}_n(\mathbb{C}); V)$ with coefficients in an S_n-rep V, it suffices to understand the action of S_n on $\mathrm{PConf}_n(\mathbb{C})$.

Remark There are equivalent, more geometric ways to define (co)homology with local coeff. Let M be a discrete $\mathbb{Z}[\pi_1]$-module.

If $M \longrightarrow E \longrightarrow X$ is a bundle with fibres

M and monodromy defined by the action $\pi_1 \curvearrowright M$, then we can define a chain complex with cells in X labelled by a choice of lift to E.

This chain complex also computes $H_\bullet(X; M)$; see (eg) Hatcher's Algebraic Topology 3H for details.

Now for our next ingredient, the transfer map.

<u>Transfer</u>

Let X be path-connected.

$$\begin{array}{c} X' \\ p \big\downarrow \\ X \end{array} \quad \text{m-sheeted cover} \qquad m < \infty.$$

The covering map p induces a map on chain complexes:

$$\begin{array}{ccc} C_n(X') & & \sigma \\ \big\downarrow p_\# & & \big\downarrow \\ C_n(X) & & p \circ \sigma \end{array}$$

but there is also a map τ (the transfer map) in the opposite direction:

$$\begin{array}{ccc} C_n(X') & & \sum \tilde{\sigma} \quad \text{all lifts } \tilde{\sigma} \text{ of } \sigma \\ \big\uparrow \tau & & \big\uparrow \\ C_n(X) & & \sigma \end{array}$$

The sum $\sum \tilde{\sigma}$ is well-defined by our assumption that $m < \infty$.

The map τ induces "wrong-way" maps on (co)homology

$$\tau_* : H_k(X) \longrightarrow H_k(X')$$
$$\tau^* : H^k(X') \longrightarrow H^k(X)$$

Observe that $p_\# \tau$ is multiplication by m:

$$\begin{tikzcd} C_n(X') \arrow[d, shift left, "p_\#"] \\ C_n(X) \arrow[u, shift left, "\tau"] \end{tikzcd} \qquad \begin{tikzcd} & \Sigma \tilde{\sigma} \arrow[dr, mapsto, "p_\#"] & \\ \sigma \arrow[ur, mapsto, "\tau"] & & \Sigma p \circ \tilde{\sigma} = m\sigma \end{tikzcd}$$

This implies that $\tau^* p^* =$ multiplication by m

$$\begin{tikzcd} H^*(X') \arrow[d, shift right, "\tau^*"'] \\ H^*(X) \arrow[u, shift right, "p^*"'] \end{tikzcd}$$

so $\ker(p^*) \subseteq$ m-torsion in $H^*(X)$.

and p^* injects away from m-torsion.

Thus, "up to m-torsion", we have containment of $H^*(X)$ in $H^*(X')$.

<u>Exercise</u> (not-too-difficult)

Consider a finite cover $|\Gamma| = m$

$$\begin{tikzcd} X' \arrow[loop right, "\Gamma"] \arrow[d] \\ X = X'/\Gamma \end{tikzcd}$$

and field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$ or $\mathrm{char}(\mathbb{F}) \nmid m$.

Then

- $p^* : H^k(X; \mathbb{F}) \hookrightarrow H^k(X'; \mathbb{F})$ injects
- $\mathrm{Im}(p^*) = H^k(X'; \mathbb{F})^\Gamma$

↑ subspace ptwise fixed by Γ.

<u>Transfer with twisted coefficients.</u>

We want an analogous identification for (co)homology with twisted coefficients.

Fortunately, in our $X = \mathrm{Conf}_n$ case we are interested in

$\mathbb{C}$ reps of finite groups. $\mathbb{C}S_n$ is semi-simple and the situation is straight-forward:

tensoring over $\mathbb{C}S_n$ is exact, so

$$H_\bullet(X;V) := H_\bullet(C_\bullet(X') \otimes_{\mathbb{C}S_n} V)$$
$$\cong H_\bullet(X';\mathbb{C}) \otimes_{\mathbb{C}S_n} V$$

Similarly $H^\bullet(X;V) \cong H_\bullet(X';\mathbb{C}) \otimes_{\mathbb{C}S_n} V$

Notably $\dim_{\mathbb{C}} H^\bullet(\mathrm{Conf}_n(\mathbb{C});V)$
$$= \langle H^\bullet(\mathrm{PConf}_n(\mathbb{C});\mathbb{C}) ; V \rangle_{S_n}$$
the inner product on characters of $\mathbb{C}S_n$-reps (with orthogonal basis the irreducible reps).

This last equality uses the fact that S_n-reps are self-dual.

Upshot: we need to study the S_n-rep $H^i(\mathrm{PConf}_n(\mathbb{C}))$.

Next we will introduce Étale cohomology, and its advent to solve the Weil conjectures.

<u>Weil Conjectures.</u> ← Much-celebrated conjectures of André Weil

<u>Defn</u> Let V be a non-singular projective algebraic variety over $\mathbb{F}_q$.

A <u>local zeta function</u> is a particular form of generating function encoding point-counts for V:

$$\zeta(V,s) = \exp\left(\sum \frac{N_k}{k}(q^{-s})^k\right)$$

$N_k = \#\ \mathbb{F}_{q^k}$ - points of V

The Weil Conjectures (1949 - since proven)

① ζ will be a rational function of $T = q^{-s}$

$$\frac{P_1(T) \cdots P_{2n-1}(T)}{P_0(T) \cdots P_{2n}(T)}$$

(proven Dwork, 1960)

② ζ satisfy certain functional equations (involving the Euler characteristic of V)

(proven Grothendieck, 1965) These imply certain relations between the roots of P_i.

③ The roots of P_i lie in restricted places (mirrors the Riemann Hypothesis)

(proven for $\zeta(V,s)$ by Deligne, 1974)

④ Suppose V is defined by polynomials with coeffs in $\mathbb{Z}$, so it is possible to take $\mathbb{F}_q$-points $V_{\mathbb{F}_q}$ or complex points $V_{\mathbb{C}}$.

Then $\deg(P_i) = i^{th}$ Betti number of $V_{\mathbb{C}}$

Weil proposed that these conjectures could be solved by defining an appropriate cohomology theory for $V_{\mathbb{F}_{q^m}}$.

(since $V_{\mathbb{F}_{q^m}}$ is discrete, our usual singular/cellular cohomology theories do not capture its structure)

Grothendieck (with input from Artin, Deligne, ...) developed Étale cohomology

$H^i_{et}(V_{\mathbb{F}_{q^m}}; \mathbb{Q}_\ell)$ coefficients in ℓ-adics $\mathbb{Q}_\ell$, $\ell \neq q$.

In same sense the role of open sets is replaced by that of "étale morphisms", and fundamental groups by Galois groups.

These étale cohomology groups have:

- $H^i_{et}(X;\mathbb{Q}_\ell)$ are finite-dim $\mathbb{Q}_\ell$ vector spaces, vanishing for $i > 2\dim(X)$ and below $i=0$.
- Poincaré duality $\quad H^i(X) \times H^{n-i}(X) \longrightarrow H^n(X)$
- Kunneth
- action of Frobenius & Galois group
- relationship to $H^\cdot(X_{\mathbb{C}};\mathbb{C})$
- Lefschetz fixed pt formula.

Expanding on the last bullet point, recall the classic Lefschetz fixed-pt theorem

Y cpt top space (say, smooth mfd)

$f \curvearrowright Y$ with isolated fixed points.

$$\sum_{z \in Fix(f)} \mathrm{index}(z) = \sum (-1)^i \,\mathrm{Trace}\left\{ f : H_i(Y;\mathbb{C}) \right\}$$

For Y not cpt, we replace $H_i(Y;\mathbb{C})$ with cptly-supported homology, which we can identify with Poincaré dual $H^{n-i}(Y;\mathbb{C})$. $(n = \dim Y)$.

There is an analogous Grothendieck-Lefschetz fixed pt theorem:

$$\#\mathrm{Fix}(\mathrm{Frob}_q^m) = |X_{\mathbb{F}_{q^m}}| = \sum_{i \geq 0} (-1)^i \,\mathrm{Tr} : \left\{ \mathrm{Frob}_q^m \curvearrowright H^{i\,\mathrm{dual}}_{et}(X_{\overline{\mathbb{F}_q}};\mathbb{Q}_\ell) \right\}$$

This formula allowed Grothendieck to express the point counts, and hence the zeta functions, in terms of the étale cohomology groups.

Artin's Comparison Thm

Under suitably nice conditions,

$$H^i_{et}(X_{\mathbb{F}_q}; \mathbb{Q}_\ell) \xrightarrow{\cong} H^i(X_{\mathbb{C}}; \mathbb{Q}_\ell)$$

Specializing now to $X = \mathrm{Conf}_n$,
It is known that $\mathrm{Frob}_q \curvearrowright H^i_{et}(X; \mathbb{Q}_\ell)$ by multiplication by q^i

so $|X_{\mathbb{F}_q}|$

$$= \sum_{i \geq 0} (-1)^i \,\mathrm{Trace}\left\{ \mathrm{Frob}_q \curvearrowright H^{i\text{-dual}}_{et}(X_{\overline{\mathbb{F}_q}}; \mathbb{Q}_\ell) \right\}$$

$$= \sum_{i \geq 0} (-1)^i q^{n-i} \dim_{\mathbb{Q}_\ell} H^{i\text{ dual}}_{et}(X_{\overline{\mathbb{F}_q}}; \mathbb{Q}_\ell)$$

$$= \sum_{i \geq 0} (-1)^i q^{n-i} \dim_{\mathbb{C}} H^i(X_{\mathbb{C}}; \mathbb{C})$$

and we recover the formula

$$|\mathrm{Conf}_n / \mathbb{F}_q| = q^n - q^{n-1}$$ from the intro.

There is moreover a twisted version of the Grothendieck-Lefschetz formula.

Fix V_n an S_n-rep with character χ

Recall that for any $f \in \mathrm{Fix}(\mathrm{Frob}_q \curvearrowright \mathrm{Conf}_n(\overline{\mathbb{F}_q}))$, we could associate a conj class $\sigma_f \in S_n$.

Then

$$\sum_{f \in Fix(Frob_q)} \chi(\sigma_f) = \sum_{i \geq 0} (-1)^i q^{n-i} \dim_{\mathbb{Q}_\ell} H^{i, dual}_{et}(conf; V_n)$$

$$= \sum_{i \geq 0} (-1)^i q^{n-i} \langle H^i(PConf_n \mathbb{C}; \mathbb{C}); V \rangle_{S_n}.$$

<u>Implications of rep stability</u>:

Define $c_i : \coprod_{n \geq 0} S_n \longrightarrow \mathbb{Z}$

$c_i : \sigma \longmapsto$ # i-cycles in cycle type of σ

These class functions are algebraically independent on $\coprod S_n$,

and we call elements $P \in \mathbb{Q}[c_1, c_2, \ldots]$ <u>character polynomials</u> for S_n. $\deg(c_i) := i$.

P defines a (virtual) character of S_n $\forall n$.

<u>Thm</u> (Church-Ellenberg-Farb)

$H^i(PConf_n \mathbb{C}; \mathbb{C})$ is representation stable as a sequence of S_n-reps.

This implies, in particular, for any character polynomial P,

$$\langle H^i(PConf_n(\mathbb{C}); \mathbb{C}); P \rangle_{S_n}$$

is constant in n once $n > 2i + \deg(P)$

Thus these statistics "stabilize" as $n \longrightarrow \infty$.

$$\sum_{f \in \mathrm{Fix}(\mathrm{Frob}_q)} P(\sigma_f) = \sum (-1)^i q^{n-i} \langle H^i(\mathrm{PConf}_n; \mathbb{C}), V_P \rangle$$

P with c_i evaluated at # irreducible deg i factors of f

(Formula $*$)

The first entry in our table on page 1 comes from $P = c_1$, which counts 1-cycles (and so $P(\sigma_f) = $ # linear factors in f).

$P = c_1$ gives the characters of the canonical S_n-reps $S_n \curvearrowright \mathbb{C}^n$, since trace = # fixed points.

So we get

$$\sum_f \#\text{ linear factors} = \sum (-1)^i q^{n-i} \langle H^i(\mathrm{PConf}_n; \mathbb{C}); \mathbb{C}^n \rangle_{S_n}$$

We can divide through by $|\mathrm{Fix}(\mathrm{Frob}_q)| = q^n - q^{n-1}$ to get the expected number of linear factors of a random poly f over $\mathbb{F}_q$.

– or more generally, the expected value of the polynomial p on a random polynomial $f(t)$.

The stability results of Church-Ellenberg-Farb imply that the normalized formula $(*)$ converges in the limit as $n \longrightarrow \infty$.