

Braid groups, representation stability, and polynomials over  $\mathbb{F}_q$ .

The goal of this talk is to describe a (surprising) relationship between the structure of the cohomology of the braid group  $B_n$  with local coefficients in  $S_n$ -representations, and certain statistics for polynomials over finite fields.

Specifically, let  $X$  denote the space of monic square-free  $\deg n$  polynomials with coefficients in  $\mathbb{F}_q$ . Let  $P$  be a polynomial in the number of irreducible factors of  $f \in X$  of given lengths. There is a way to associate an  $S_n$ -rep  $V$  over  $\mathbb{C}$  to  $P$ , and

$$\sum_{f \in X} P(f) = \sum_{i \geq 0} (-1)^i q^{n-i} \dim_{\mathbb{C}} H^i(B_n; V)$$

where  $H^{n-i}(B_n; V)$  is the cohomology of  $B_n$  with twisted coefficients in  $V$ . (All this terminology will be defined).

This talk is based on a paper by Church-Ellenberg-Farb, "Representation Stability in cohomology and asymptotics for families of varieties over finite fields"

ArXiv: 1309.6038

In this talk, we will

- ① Recall the definition of configuration space and its relationship to the (pure) braid group and moduli spaces of polynomials.
- ② Review the definitions of (co)homology with local coefficients, and the transfer maps.
- ③ Overview Étale cohomology and the Grothendieck-Lefschetz fixed point theorem.
- ④ See implications of rep stability to polynomial statistics.

Some useful formulas:

Let  $X =$  space of monic deg  $n$  polynomials with coefficients in  $\mathbb{F}_q$  ( $q$  prime)

$B_n =$  Artin's braid group

Statistic on  $X$

Count

Cohomology of  $B_n$  with local coeffs.

Ranks of  $i$ th cohomology groups

$\#|X|$

$q^n - q^{n-1}$

$H^*(B_n; \mathbb{C})$

$\beta_0 = \beta_1 = 1$   
 $\beta_i = 0, i > 1$

$\sum_{f \in X} \# \text{ linear factors of } f$

$q^n - 2q^{n-1} + 2q^{n-2} \dots \pm 2q^2 \pm q$

$H^*(B_n; \mathbb{Q}^n)$

$\beta_0 = \beta_{n-1} = 1$   
 $\beta_i = 2, i=1, \dots, n-2$   
 $\beta_i = 0, i \geq n$

$\sum_{f \in X} P(f)$

$\sum_{i \geq 0} (-1)^i \beta_i q^{n-i}$

$H^*(B_n; V_p)$

$\beta_i$

$P$  poly in  $\# \text{ deg } k$  irreducible factors of  $f$ ,  $k=1, \dots, N$  for some  $N$ .



Cohomology of  $B_n$  with coeffs in  $S_n$ -rep  $V_p$  whose character is (same as) determined by  $P$ .

Today's Main objects: Conf<sub>n</sub> and B<sub>n</sub> Choose a field IF.

Let  $PConf_n(IF) = \{ (x_1, \dots, x_n) \in IF^n \mid x_i \neq x_j \text{ for } i \neq j \}$   
denote the pure (ordered) configuration space of distinct points in IF.

For  $IF = \mathbb{C}$ , we topologize  $PConf_n(\mathbb{C})$  as a subspace of  $\mathbb{C}^n$ .

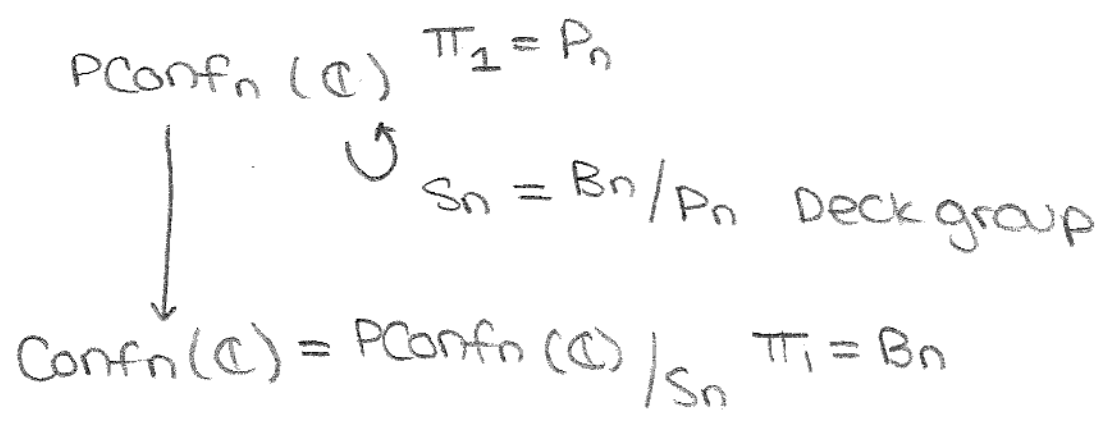
$PConf_n(\mathbb{C})$  is a  $K(\pi, 1)$ -space for its fundamental group, the pure braid group  $P_n$ .

The symmetric group  $S_n \curvearrowright PConf_n(IF)$  by permuting the  $n$ -tuples.

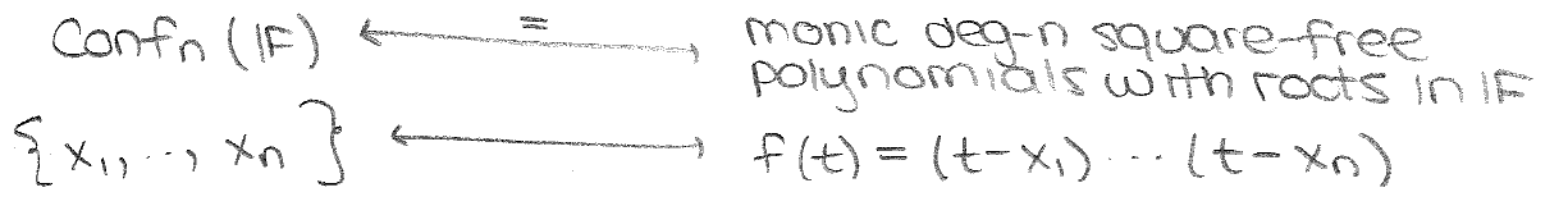
The quotient  $Conf_n(IF) := PConf_n(IF) / S_n$  is the (unordered) configuration space of sets of  $n$  elements in IF.

$Conf_n(\mathbb{C})$  is a  $K(\pi, 1)$  for Artin's braid group  $B_n$ .

We thus have a normal covering map:



For a field IF, we can identify



This space of polynomials is an algebraic variety, the complement of the vanishing set of the discriminant in coeffs of f.

## Frobenius morphism.

Given a variety defined by polynomials over  $\mathbb{F}_q$ , ( $q$  prime) the Frobenius map acts on its  $\overline{\mathbb{F}_q}$  points.

For  $x \in \overline{\mathbb{F}_q}$ ,  $\text{Frob}_q: x \mapsto x^q$ .

Frobenius fixes precisely the  $\mathbb{F}_q$  points

(since  $\text{Frob}_q(x) = x \iff x$  is a root of  $t^q - t$ ).

More generally, the  $m$ th iterate  $\text{Frob}_q^m$  fixes the points in  $\mathbb{F}_{q^m}$ . These iterates generate the Galois groups of field extensions of  $\mathbb{F}_q$ .

In our case,  $\text{Frob}_q \simeq \text{Conf}_n(\overline{\mathbb{F}_q})$ .

A polynomial  $f \in \text{Conf}_n(\overline{\mathbb{F}_q})$  is fixed by  $\text{Frob}_q$  exactly when its coefficients lie in  $\mathbb{F}_q$ .

In this case,  $\text{Frob}_q$  permutes the roots of  $f(t)$ .

The orbits of roots under  $\text{Frob}_q$  correspond to the irreducible factors (over  $\mathbb{F}_q$ ) of  $f$ .

Thus, for each  $f \in \text{Fix}(\text{Frob}_q)$

$\text{Frob}_q$  determines a permutation  $\sigma_f \in S_n$  on the roots of  $f$ .

(The roots are unordered, so  $\sigma_f$  is defined up to conjugacy.)

More on this soon... first, a review of local coefficients.

## (co)homology with local (twisted) coefficients.

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Let  $X$  be a path-connected topological space,  
with  $\pi_1 = \pi_1(X)$ .

$\begin{array}{c} \tilde{X} \\ \downarrow \\ X \end{array} \quad \begin{array}{c} \hookrightarrow \\ \pi_1 \end{array} \quad \pi_1 \text{ acts on the universal cover } \tilde{X} \text{ of } X \\ \text{by Deck transformations.}$

This induces an action  $\pi_1 \curvearrowright C_n(\tilde{X})$   
on the singular chains on  $\tilde{X}$ .  
This action commutes with the bdy maps.

Let  $M$  be any left  $\mathbb{Z}[\pi_1]$ -module.

Define  $C_n(X; M) := C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1]} M$

$C^n(X; M) := \text{Hom}_{\mathbb{Z}[\pi_1]}(C_n(\tilde{X}); M)$

The boundary maps  $\partial$  on  $C_n(\tilde{X})$  induce maps on  
 $C_n(X; M)$  and  $C^n(X; M)$  making these (co)-chain  
complexes.

Def<sup>n</sup>  $H_*(X; M) := H_*(C_*(X; M))$

$H^*(X; M) := H^*(C^*(X; M))$

The homology of these complexes is what we  
define to be the (co)-homology groups of  $X$  with  
twisted (or local) coefficients in  $M$ .

Remark In order to form the tensor product  
 $C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1]} M$ , we need to turn  $C_n(\tilde{X})$   
from a left to a right  $\mathbb{Z}[\pi_1]$ -module.

To do this we define:  $g \cdot \sigma := \sigma \cdot g^{-1}$

Thus tensoring over  $\mathbb{Z}[\pi_1]$  is tantamount to  
tensoring over  $\mathbb{Z}$  and madding out by the diagonal

action of  $\pi_1$ :

$$C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1]} M = \left( \langle \sigma \otimes m \rangle / \sigma \otimes m = \sigma g^{-1} \otimes gm \right)$$

First examples of (co)homology with twisted coefficients:

Eg  $M = \mathbb{Z}$  trivial  $\mathbb{Z}[\pi_1]$ -module.

$$\begin{aligned} \text{Then } C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1]} \mathbb{Z} &\cong C_n(\tilde{X}) / \text{action of } \pi_1 \\ &\cong C_n(X) \end{aligned}$$

since we can identify chains in  $C_n(X)$  with  $\pi_1$ -orbits of chains in  $C_n(\tilde{X})$

so  $H_*(X; \mathbb{Z})$  is the usual homology of  $X$  with integer coefficients.

Exercise  $H^*(X; \mathbb{Z})$  is usual cohomology of  $X$ .

Eg  $M = \mathbb{Z}[\pi_1]$  group ring.

$$\text{Then } C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1]} \mathbb{Z}[\pi_1] \cong C_n(\tilde{X})$$

$$\text{and } H_*(X; \mathbb{Z}[\pi_1]) \cong H_*(\tilde{X}; \mathbb{Z})$$

Eg  $M = \mathbb{Z}[\pi_1 / \pi_1']$  for  $\pi_1' \triangleleft \pi_1$

$$\text{Then } C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1]} \mathbb{Z}[\pi_1 / \pi_1']$$

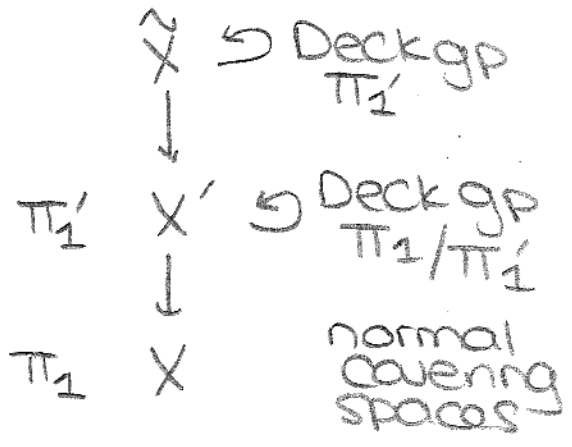
$$\cong C_n(\tilde{X}) / \pi_1' \cong C_n(X')$$

$$\text{and } H_*(X, M) \cong H_*(X'; M)$$

↑ normal cover assoc. to  $\pi_1'$

$$\begin{array}{ccc} \tilde{X} & & \\ \downarrow & & \\ X' & \pi_1' & \\ \downarrow & & \\ X & \pi_1 & \end{array}$$

More generally, if  $M$  is a  $\mathbb{Z}[\pi_1]$ -module such that some normal subgroup  $\pi_1' \triangleleft \pi_1$  acts trivially on  $M$  (ie, the action  $\pi_1 \curvearrowright M$  factors through  $\pi_1/\pi_1'$ )

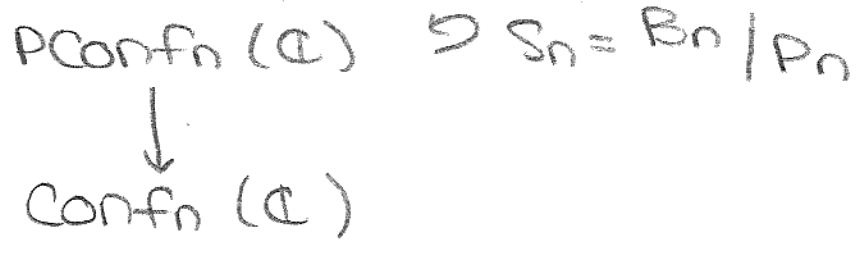


Then

$$\begin{aligned}
 C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1]} M \\
 \cong C_n(X') \otimes_{\mathbb{Z}[\pi_1/\pi_1']} M
 \end{aligned}$$

Upshot: To understand  $H_*(X; M)$  for a  $\mathbb{Z}[\pi_1/\pi_1']$ -module  $M$ , it suffices to understand the action of  $\pi_1/\pi_1'$  on the cover  $X'$ .

In our case:



To understand  $H_*(\text{Conf}_n(\mathbb{C}); V)$  and  $H^*(\text{Conf}_n(\mathbb{C}); V)$  with coefficients in an  $S_n$ -rep  $V$ , it suffices to understand the action of  $S_n$  on  $\text{PConf}_n(\mathbb{C})$ .

Remark There are equivalent, more geometric ways to define (co)homology with local coeff. Let  $M$  be a discrete  $\mathbb{Z}[\pi_1]$ -module. If  $M \rightarrow E \rightarrow X$  is a bundle with fibres

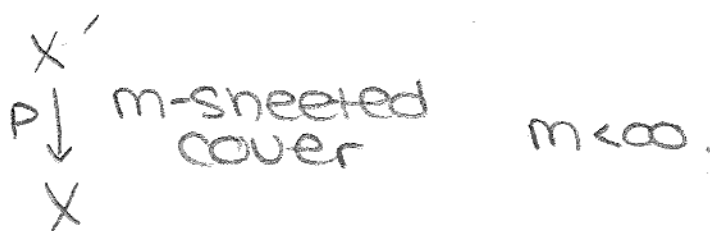
$M$  and monodromy defined by the action  $\pi_1 \mathbb{R} \curvearrowright M$ , then we can define a chain complex with cells in  $X$  labelled by a choice of lift to  $E$ .

This chain complex also computes  $H_*(X; M)$ ; see (eg) Hatcher's Algebraic Topology 3H for details.

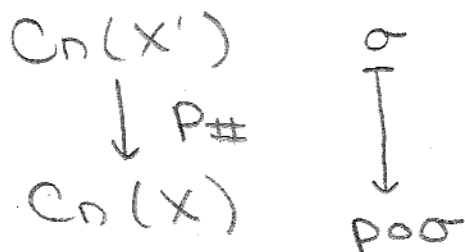
Now for our next ingredient, the transfer map.

### Transfer

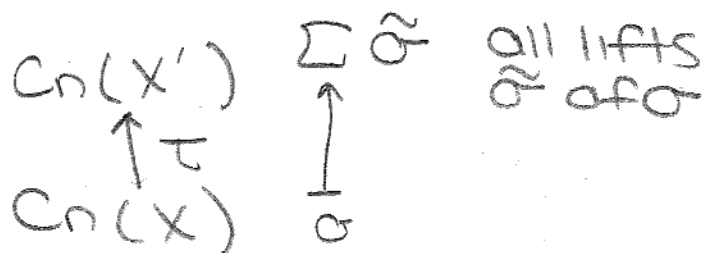
Let  $X$  be path-connected.



The covering map  $p$  induces a map on chain complexes:



but there is also a map  $\tau$  (the transfer map) in the opposite direction:



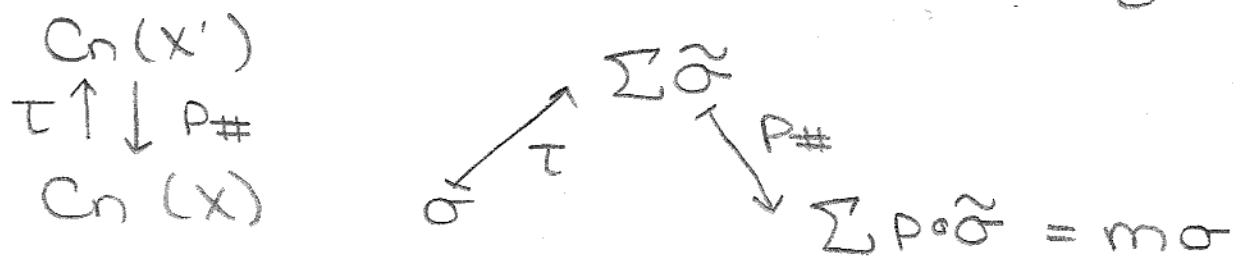
The sum  $\sum \tilde{\sigma}$  is well-defined by our assumption that  $m < \infty$ .

The map  $\tau$  induces "wrong-way" maps on (co)homology

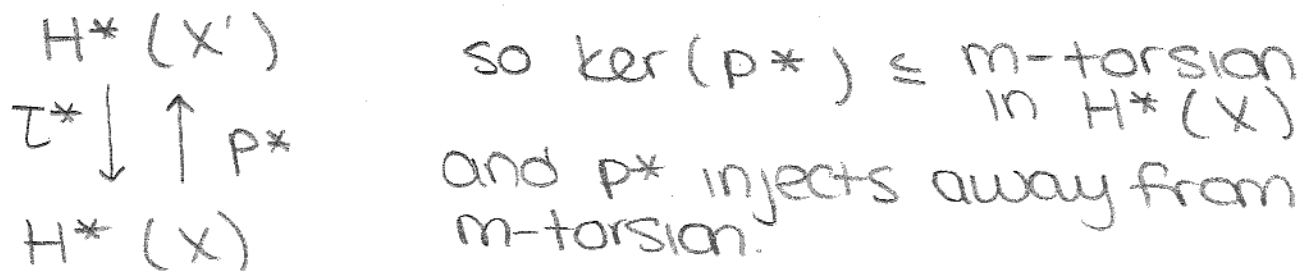
$$\begin{aligned}
 \tau_* &: H_k(X) \longrightarrow H_k(X') \\
 \tau^* &: H_k(X') \longrightarrow H_k(X)
 \end{aligned}$$



Observe that  $P_{\#}\tau$  is multiplication by  $m$ :



This implies that  $\tau^*p^* = \text{multiplication by } m$



Thus, "up to  $m$ -torsion", we have containment of  $H^*(X)$  in  $H^*(X')$ .

Exercise (not-too-difficult)

Consider a finite cover  $X' \xrightarrow{\Gamma} X = X'/\Gamma$  and field  $\mathbb{F}$  with  $\text{char}(\mathbb{F}) = 0$  or  $\text{char}(\mathbb{F}) \nmid m$ .

- Then
- $p_* : H^k(X; \mathbb{F}) \hookrightarrow H^k(X'; \mathbb{F})$  injects
  - $\text{Im}(p_*) = H^k(X; \mathbb{F})^{\Gamma}$   
 $\uparrow$  subspace ptwise fixed by  $\Gamma$ .

Transfer with twisted coefficients.

We want an analogous identification for (co)homology with twisted coefficients.

Fortunately, in our  $X = \text{Conf}_n$  case we are interested in

$\mathbb{C}$  reps of finite groups.  $\mathbb{C}S_n$  is semi-simple and the situation is straightforward:

tensoring over  $\mathbb{C}S_n$  is exact, so

$$\begin{aligned} H_*(X; V) &:= H_*(C_*(X') \otimes_{\mathbb{C}S_n} V) \\ &\cong H_*(X'; \mathbb{C}) \otimes_{\mathbb{C}S_n} V \end{aligned}$$

Similarly  $H^*(X; V) \cong H^*(X'; \mathbb{C}) \otimes_{\mathbb{C}S_n} V$

$$\begin{aligned} \text{Notably } \dim_{\mathbb{C}} H^*(\text{Conf}_n(\mathbb{C}); V) \\ = \langle H^*(\text{PConf}_n(\mathbb{C}); \mathbb{C}); V \rangle_{S_n} \end{aligned}$$

the inner product on characters of  $\mathbb{C}S_n$ -reps (with orthogonal basis the irreducible reps).

This last equality uses the fact that  $S_n$ -reps are self-dual.

upshot: we need to study the  $S_n$ -rep  $H^i(\text{PConf}_n(\mathbb{C}))$ .

Next we will introduce Étale cohomology, and its advent to solve the Weil conjectures.

Weil Conjectures. ← Much-celebrated conjectures of André Weil

Defn Let  $V$  be a non-singular projective algebraic variety over  $\mathbb{F}_q$ .

A local zeta function is a particular form of generating function encoding point-counts for  $V$ :

$$\zeta(V, s) = \exp \left( \sum \frac{N_k}{k} (q^{-s})^k \right)$$

$N_k = \# \mathbb{F}_{q^k}$ -points of  $V$ .

The Weil Conjectures (1949 - since proven)

①  $\zeta$  will be a rational function of  $T = q^{-s}$

$$\frac{P_1(T) \cdots P_{2n-1}(T)}{P_0(T) \cdots P_{2n}(T)}$$

(proven Dwork, 1960)

②  $\zeta$  satisfy certain functional equations  
(involving the Euler characteristic of  $V$ )

(proven Grothendieck, 1965) These imply certain relations between the roots of  $P_i$ .

③ The roots of  $P_i$  lie in restricted places.  
(mirrors the Riemann Hypothesis)

(proven for  $\zeta(V, s)$  by Deligne, 1974)

④ Suppose  $V$  is defined by polynomials with coeffs in  $\mathbb{Z}$ , so it is possible to take  $\mathbb{F}_q$ -points  $V/\mathbb{F}_q$  or complex points  $V_{\mathbb{C}}$ .

Then  $\deg(P_i) = i$ th Betti number of  $V_{\mathbb{C}}$

Weil proposed that these conjectures could be solved by defining an appropriate cohomology theory for  $V/\mathbb{F}_q$ .  
(since  $V/\mathbb{F}_q$  is discrete, our usual singular/cellular cohomology theories do not capture its structure)

Grothendieck (with input from Artin, Deligne, ...) developed Étale cohomology

$$H^i_{\text{ét}}(V/\mathbb{F}_q; \mathbb{Q}_\ell) \quad \text{coefficients in } \ell\text{-adics } \mathbb{Q}_\ell, \ell \neq q.$$

In some sense the role of open sets is replaced by that of "étale morphisms", and fundamental groups by Galois groups.

These étale cohomology groups have:

- $H^i_{\text{ét}}(X; \mathbb{Q}_\ell)$  are finite-dim  $\mathbb{Q}_\ell$  vector spaces, vanishing for  $i > 2 \dim(X)$  and below  $i=0$ .
- Poincaré duality  $H^i(X) \times H^{n-i}(X) \rightarrow H^n(X)$
- Kunneth
- action of Frobenius & Galois group
- relationship to  $H^*(X_{\mathbb{C}}; \mathbb{C})$
- Lefschetz fixed pt formula.

Expanding on the last bullet point, recall the classic Lefschetz fixed-pt theorem

$Y$  cpt top space (say, smooth mfd)

$f: Y \rightarrow Y$  with isolated fixed points.

$$\sum_{z \in \text{Fix}(f)} \text{index}(z) = \sum (-1)^i \text{Trace} \{ f: H_i(Y; \mathbb{C}) \}$$

For  $Y$  not cpt, we replace  $H_i(Y; \mathbb{C})$  with cptly-supported homology, which we can identify with Poincaré dual  $H^{n-i}(Y; \mathbb{C})$ . ( $n = \dim Y$ ).

There is an analogous Grothendieck-Lefschetz fixed pt theorem:

$$\# \text{Fix}(\text{Frob}_q^m) = |X_{\mathbb{F}_{q^m}}| = \sum_{i \geq 0} (-1)^i \text{Tr} : \left\{ \begin{array}{l} \text{Frob}_q^m \\ H_{\text{ét}}^{i, \text{dual}}(X_{\overline{\mathbb{F}_q}}; \mathbb{Q}_\ell) \end{array} \right\}$$

This formula allowed Grothendieck to express the point counts, and hence the zeta functions, in terms of the étale cohomology groups.

### Artin's Comparison Thm

Under suitably nice conditions,

$$H_{\text{et}}^i(X/\mathbb{F}_q; \mathbb{Q}_\ell) \xrightarrow{\cong} H^i(X_{\mathbb{C}}; \mathbb{Q}_\ell)$$

Specializing now to  $X = \text{Conf}_n$ ,

It is known that  $\text{Frob}_q \curvearrowright H_{\text{et}}^i(X; \mathbb{Q}_\ell)$  by multiplication by  $q^i$

so  $|X/\mathbb{F}_q|$

$$\begin{aligned}
&= \sum_{i \geq 0} (-1)^i \text{Trace} \left\{ \text{Frob}_q \curvearrowright H_{\text{et}}^{i\text{-dual}}(X/\overline{\mathbb{F}}_q; \mathbb{Q}_\ell) \right\} \\
&= \sum_{i \geq 0} (-1)^i q^{n-i} \dim_{\mathbb{Q}_\ell} H_{\text{et}}^{i\text{-dual}}(X/\overline{\mathbb{F}}_q; \mathbb{Q}_\ell) \\
&= \sum_{i \geq 0} (-1)^i q^{n-i} \dim_{\mathbb{C}} H^i(X_{\mathbb{C}}; \mathbb{C})
\end{aligned}$$

and we recover the formula

$$|\text{Conf}_n/\mathbb{F}_q| = q^n - q^{n-1} \text{ from the intro.}$$

There is moreover a twisted version of the Grothendieck-Lefschetz formula.

Fix  $V_n$  an  $S_n$ -rep with character  $\chi$

Recall that for any  $f \in \text{Fix}(\text{Frob}_q \curvearrowright \text{Conf}_n(\overline{\mathbb{F}}_q))$ ,

we could associate a conj class  $\sigma_f \in S_n$ .

Then

$$\begin{aligned} \sum_{f \in \text{Fix}(\text{Frob}_q)} \chi(\sigma_f) &= \sum_{i \geq 0} (-1)^i q^{n-i} \dim_{\mathbb{Q}_\ell} H_{\text{et}}^{i, \text{dual}}(\text{conf}; V_n) \\ &= \sum_{i \geq 0} (-1)^i q^{n-i} \langle H^i(\text{Pconf}_n \mathbb{C}; \mathbb{C}); \chi \rangle_{S_n}. \end{aligned}$$

### Implications of rep stability:

Define  $c_i : \coprod_{n \geq 0} S_n \longrightarrow \mathbb{Z}$

$c_i : \sigma \longmapsto \# \text{ } i\text{-cycles in cycle type of } \sigma$

These class functions are algebraically independent on  $\coprod S_n$ ,

and we call elements  $P \in \mathbb{Q}[c_1, c_2, \dots]$  character polynomials for  $S_n$ .  $\deg(c_i) := i$ .

$P$  defines a (virtual) character of  $S_n \forall n$ .

### Thm (Church-Eilenberg-Farb)

$H^i(\text{Pconf}_n \mathbb{C}; \mathbb{C})$  is representation stable as a sequence of  $S_n$ -reps.

This implies, in particular, for any character polynomial  $P$ ,

$\langle H^i(\text{Pconf}_n \mathbb{C}; \mathbb{C}); P \rangle_{S_n}$   
is constant in  $n$  once  $n > 2i + \deg(P)$ .

Thus these statistics "stabilize" as  $n \rightarrow \infty$ . (8)

$$\sum_{f \in \text{Fix}(\text{Frob}_q)} P(\sigma_f) = \sum (-1)^i q^{n-i} \langle H^i(\text{PCan}_f; \mathbb{C}), V_P \rangle$$

$P$  with  $c_i$  evaluated  
at # irreducible deg  $i$   
factors of  $f$

(Formula \*)

The first entry in our table on page 1 comes from  $P = c_1$ , which counts 1-cycles (and so  $P(\sigma_f) = \#$  linear factors in  $f$ ).

$P = c_1$  gives the characters of the canonical  $S_n$ -reps  $S_n \curvearrowright \mathbb{C}^n$ , since trace = # fixed points.

So we get

$$\sum_f \# \text{ linear factors} = \sum (-1)^i q^{n-i} \langle H^i(\text{PCan}_f; \mathbb{C}); \mathbb{C}^n \rangle_{S_n}$$

We can divide through by  $|\text{Fix}(\text{Frob}_q)| = q^n - q^{n-1}$  to get the expected number of linear factors of a random poly  $f$  over  $\mathbb{F}_q$ .

- or more generally, the expected value of the polynomial  $p$  on a random polynomial  $f(t)$ .

The stability results of Church-Ellenberg-Farb imply that the normalized formula (\*) converges in the limit as  $n \rightarrow \infty$ .