The topology of hyperplane complements

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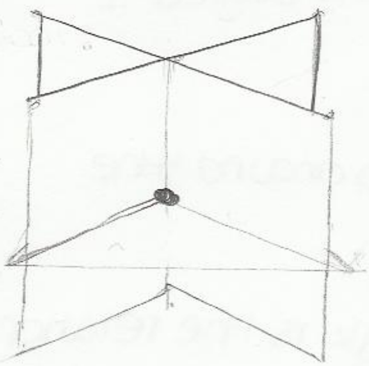
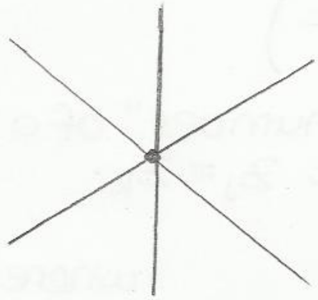
<u>Defn</u> A hyperplane arrangement A is a finite set of (codim 1) hyperplanes in a linear, affine, or projective space.

This talk will focus on linear hyperplanes (containing the origin) in a complex vector space.





Goal of study:

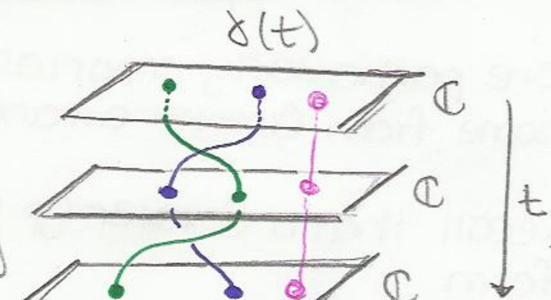
Relate the geometry / topology of the complement of A to the combinatorics of the intersection pattern of the hyperplanes.

<u>Motivating example</u>: Let $V = \mathbb{C}^n = \langle z_1, \ldots, z_n \rangle$, $A = \{ z_i = z_j \mid i \neq j \}$

The hyperplane complement $M(A) := \mathbb{C}^n \setminus \bigcup_{H \in A} H$ is precisely the ordered configuration space of $\mathbb{C}$ on n-points, the set of all ordered n-tuples of distinct complex numbers.

$\pi_1(M(A)) = PBr_n$ the pure braid group.

If we follow the n points through a loop $\gamma(t)$ in the space $M(A)$, they trace out n braided strands, each eventually returning to its own starting point.



The symmetric gp S_n acts on $\mathbb{C}^n$ by permutation matrices; its transpositions act by reflection in the planes $z_i - z_j$

The gp S_n stabilizes the set A, and acts freely on $M(A)$.

The quotient $M(A)/S_n$ is the (unordered) configuration space of n points in $\mathbb{C}$.

$\pi_1(M(A)/S_n) = Br_n$ Artin's braid group.

$$\left.\begin{array}{ll} S_n \circlearrowright M(A) & \pi_1 = PBr_n \\ \quad\quad \big\downarrow & \\ \quad M(A)/S_n & \pi_1 = Br_n \end{array}\right\} \text{normal covering space}$$

Fox-Neuwirth (1962) and Arnold (1968) studied $M(A)$.

Key topological results:

- $M(A)$ is a $K(P_n, 1)$-space
- $M(A)/S_n$ is homologically stable
- $H^*(M(A); \mathbb{Z})$ is free abelian
- $H^*(M(A); \mathbb{C})$ is generated (as an algebra) by the degree-1 classes $w_{jk} := \frac{1}{2\pi i}\left(\frac{dz_j - dz_k}{z_j - z_k}\right)$

  w_{jk} measures the "winding number" of a loop around the "hole" left by the hyperplane $z_j = z_k$.
- As an algebra,

$$H^*(M(A); \mathbb{Z}) \cong \Lambda\langle w_{ij}\rangle / R_{ijk}$$

where R_{ijk} is the relation $w_{ij}w_{jk} + w_{jk}w_{ki} + w_{ki}w_{ij}$

Goal: Generalize these results to other hyperplane complements.

Further examples: Coxeter arrangements.

One particularly important class of hyperplane complements come from Coxeter arrangements

Recall that a Coxeter gp G is a gp with a presentation of the form

$$G = \langle s_1, s_2, \dots, s_n \mid s_i^2, (s_i s_j)^{m_{ij}} \rangle \quad m_{ij} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$$

generators are involutions.

and that there is a way to associate to G a real vector space $V \cong \mathbb{R}^n$ ($n = \#$generators, the rank of G) where $G \curvearrowright V$ and each generator s_i acts by a reflection in the sense that s_i fixes a hyperplane pointwise and acts on some complement of that hyperplane by (-1).

Conversely, given any hyperplane arrangement stabilized by reflections in the hyperplanes, these reflections will generate a Coxeter group.

We construct a corresponding hyperplane arrangement by tensoring with $\mathbb{C}$,

and taking $A_G = \left\{ H \;\middle|\; \begin{matrix} H \text{ hyperplane stabilized by} \\ \text{a reflection in } G \end{matrix} \right\}$

$$M(A_G) = \Big(\mathbb{C}^n - \bigcup_{H \in A_G} H\Big) \circlearrowleft G$$

↑ all reflections (not just Coxeter generators)

NB since we have complexified, the hyperplanes have real codimension 2, and their complement is connected.

Examples: classical Weyl gps (finite reflection gps)

Type	Group
A_{n-1}	$S_n = \langle s_1, .., s_{n-1} \mid s_i^2, (s_is_j)^2 \text{ if } \lvert i-j\rvert>1, (s_is_j)^3 \text{ if } \lvert i-j\rvert=1 \rangle$ symmetric gp.

$V = \mathbb{C}^{n-1} = \langle z_1, z_2, , z_n \mid z_1+z_2+\dots+z_n = 0\rangle \cong \mathbb{C}^n/\text{diagonal}$

$A_A = \{ z_i - z_j = 0 \mid i \neq j \}$

B_n/C_n $\quad B_n = \langle t, s_1, .., s_{n-1} \mid s_i^2, t^2, (s_1t)^4, (s_it)^2 \text{ for } i>1), (s_is_j)^3 \text{ if } \lvert i-j\rvert = 1, (s_is_j)^2 \text{ if } \lvert i-j\rvert > 1 \rangle$

signed perm. gp. (hyperoctahedral gp)

$V = \mathbb{C}^n$, $A_B = \{ z_i - z_j = 0, \; z_i + z_j = 0, \; z_i = 0 \mid i \neq j \}$

D_n $\quad D_n = \langle u, s_1, , s_{n-1} \mid s_i^2, u^2, (us_1)^2, (us_2)^3, (us_i)^2 \; i \geq 3, (s_is_j)^3 \text{ if } \lvert i-j\rvert = 1, (s_is_j)^2 \text{ if } \lvert i-j\rvert > 1 \rangle$

even signed perm gp (D$D_n \subseteq B_n$, take $u = ts_1t$)

$A_D = \{ z_i - z_j, \; z_i + z_j \mid i \neq j \}$

NB The space $M(\mathcal{A}_A)$ is homotopic to the configuration space of $\mathbb{C}$ given in our first example.

(In that example, project onto $M(\mathcal{A})$ along lines parallel the diagonal $z_1 = z_2 = \cdots = z_n$)

In analogy, the fundamental gps of these Coxeter arrangement complements are called pure generalized braid gps.

Thm (Brieskorn, 1971) G finite irreducible Coxeter gp.

If G has Coxeter presentation $G = \langle s_i \in S \mid s_i^2, (s_i s_j)^{m_{ij}} \rangle$

then the generalized braid gp has presentation

$$\pi_1(M(\mathcal{A}_G)/G) = \langle s_i \in S \mid (s_i s_j)^{m_{ij}} \rangle$$

First question: when are these spaces $M(\mathcal{A}_G)$ $K(\pi,1)$'s ?

Conj (Arnold, Thom, Pham) $M(\mathcal{A}_G)$ are $K(\pi,1)$'s (and hence so are the spaces $M(\mathcal{A}_G)/G$ they cover)

Brieskorn proved the conjecture in type $A_n, B_n, D_n, G_2, F_2, I_2(p)$.

General strategy: (which works in some cases)

- construct a fibration $M(\mathcal{A}_{G_n}) \longrightarrow Y_n$ (possibly $Y_n = M(\mathcal{A}_{G_{n-1}})$) by projecting onto appropriate coordinate
- use the LES on homotopy gps associated to this fibration to prove by induction that all higher htpy gps vanish.

Eg Type A_{n-1}, take map $(z_1, \dots, z_n) \longmapsto (z_1, \dots, z_{n-1})$

$$\begin{array}{ccc} \cap & & \cap \\ M(\mathcal{A}_{A_{n-1}}) & & M(\mathcal{A}_{A_{n-2}}) \end{array}$$

Type D_n, take $(z_1, \dots, z_n) \longmapsto (z_n^2 - z_1^2, \dots, z_{n-1}^2 - z_n^2)$

$$\cap$$

prove this space is aspherical by induction. $\rightarrow$ $Y_{n-1} := \left\{ (y_1, \dots, y_{n-1}) \in \mathbb{C}^{n-1} \;\middle|\; \begin{array}{l} y_i \neq 0 \\ y_i \neq y_j \end{array} \right\}$

Exercise: Fill in details of these proofs.

Thm (Deligne 1972) The spaces $M(\mathcal{A}_G)$ are $K(\pi,1)$'s for any finite reflection gp G.

Deligne proved Brieskorn's conjecture, and in fact proved something stronger: whenever $\mathcal{A}$ is a (real) hyperplane arrangement so that the connected components of $V - \bigcup_{H\in\mathcal{A}} H$ are open simplicial cones,
then the complexified complement $V_{\mathbb{C}} - \bigcup_{H\in\mathcal{A}} H_{\mathbb{C}}$ is a $K(\pi,1)$-space.

His proof adapted Tits' theory of buildings.

More recently: Charney-Davis (1991) prove that an analogous construction for infinite Coxeter gps is a $K(\pi,1)$ for almost all such gps.

Their proof uses CAT(0)-geometry to analyze a "modified Deligne complex".

Second Question: what is $H^*(M(\mathcal{A}_G);\mathbb{Z})$?

Thm (Brieskorn)

- $H^p(M(\mathcal{A}_G);\mathbb{Z})$ is free abelian. Its rank is $\#\{w\in G \mid \ell(w)=p\}$ where length ℓ is taken with respect to the set of all reflections in G.
- Suppose the hyperplanes $\mathcal{A}_G$ are given by the linear forms ℓ_j. Then $H^*(M(\mathcal{A}_G);\mathbb{C})$ is generated by the differential forms
$$\omega_j := \frac{1}{2\pi i}\frac{d\ell_j}{\ell_j}$$

The integral cohomology $H^*(M(\mathcal{A}_G);\mathbb{Z})$ is isomorphic (as an algebra) to the $\mathbb{Z}$-subalgebra of meromorphic forms on $\mathbb{C}^n$ generated by ω_j.

This theorem generalizes Arnold to all finite Coxeter groups.

Brieskorn's result uses the following lemma:

Lemma (Brieskorn) Let H_i ($i \in I$) be a set of hyperplanes in $\mathbb{C}^n$ (possibly affine)

For $0 \le p \le n$, let $I_{p,1}, \ldots, I_{p,N_p}$ be all the maximal subsets of I such that $\operatorname{codim}\left(\bigcap_{i \in I_{p,k}} H_i \right) = p$.

Then the inclusions

$$\left(\mathbb{C}^n - \bigcup_{i \in I} H_i\right) \hookrightarrow \left(\mathbb{C}^n - \bigcup_{i \in I_{p,k}} H_i\right)$$

induce isomorphisms

$$H^p\left(\mathbb{C}^n - \bigcup_{i \in I} H_i ; \mathbb{Z}\right) \cong \bigoplus_{k=1}^{N_p} H^p\left(\mathbb{C}^n - \bigcup_{i \in I_{p,k}} H_i ; \mathbb{Z}\right)$$

<u>Third Question</u>: Are the generalized braid groups hom. stable?

<u>Thm</u> (Brieskorn) Let G_n be type A_n, $B_n|C_n$, or D_n.
Let BrG_n denote the generalized braid gp associated to G_n, that is, $\pi_1(M(A_{G_n})/G_n)$.

Then $H^p(BrG_{n+1} ; \mathbb{Z}) \cong H^p(BrG_n ; \mathbb{Z})$ for $n \ge 2p+2$.

<u>Fourth Question</u>: How can we determine the algebra $H^*(M(A_G))$ from the combinatorial data of the arrangement A_G?

Orlik-Solomon (1980) extended the results of Arnold-Brieskorn-Deligne

Let A be a finite set of hyperplanes in $\mathbb{C}^n$, $M(A)$ their complement.
They give an explicit description of the algebra $H^*(M(A), \mathbb{Q})$ using the combinatorial data of the poset $L(A)$ of intersections of the hyperplanes.
They give a formula for the Poincaré polynomial $P(t) = \sum B_i t^i$ for $M(A)$ in terms of the Mobius function assoc. to $L(A)$. Betti number

If $A = A_G$ is a hyperplane arrangement associated to a Coxeter gp. G, they describe the character of g on $H^p(M(A);\mathbb{Q})$ in terms of the mobius function associated to the lattice of fixed points of g in $L(A)$.

Def^{n} Let $L(A)$ be the set of spaces of the form
$$X = H_1 \cap H_2 \cap \cdots \cap H_p \quad (H_i \in A)$$
and the space $\mathbb{C}^n$, ordered by reverse inclusion. $(x \leq y \Leftrightarrow x \supseteq y)$

$L(A)$ is a lattice with minimal elt $\mathbb{C}^n$, maximal elt $\cap H_i$ $(H_i \in A)$.
↑ poset where each pair of elts have a sup and an inf.

Def^{n} A lattice L is geometric if the following hold: (L assumed finite)

(i) For each $x \in L$, every maximal linearly ordered subset
$$0 = x_0 < x_1 < x_2 < \cdots < x_p = x$$ have the same length.
This length is called the rank $r(x) = p$ of x.
Elements of length 1 are atoms.

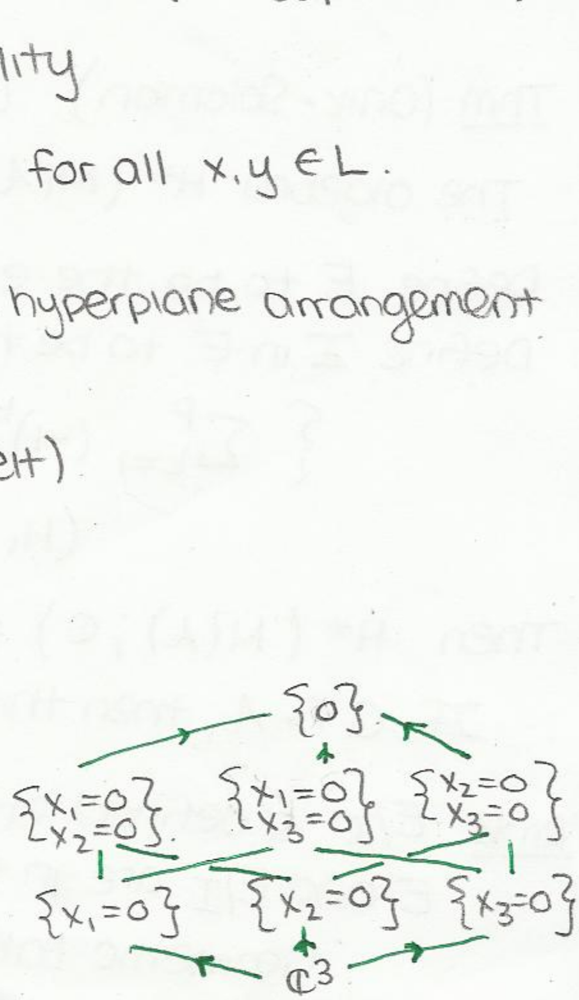
(ii) Every element in $L - \{0\}$ is a join of atoms. (join means supremum)

(iii) The rank function satisfies the inequality
$$r(x \wedge y) + r(x \vee y) \leq r(x) + r(y) \text{ for all } x, y \in L.$$

Exercise: The lattice $L(A)$ associated to a hyperplane arrangement is a geometric lattice with

- $0 = \mathbb{C}^n$ (min elt) $\quad$ • $\mathbb{1} = \bigcap_A H_i$ (max elt)
- $r(x) = \operatorname{codim}(x)$
- $x \vee y = x \cap y$ and $x \wedge y \supseteq x + y$
- atoms are the hyperplanes $H_i \in A$.

Example Let A be the set of hyperplanes
$$\{x_i = 0 \mid i = 1, 2, 3\}$$
in $\mathbb{C}^3$. Then $L(A)$ is shown here:



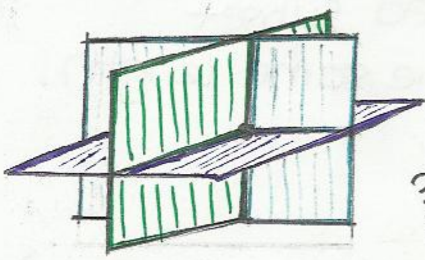
Example: Let $\mathcal{A}$ be the hyperplanes $\{x_i - x_j = 0 \mid i \neq j\}$ in $\mathbb{C}^n$.

Then $L(\mathcal{A})$ is precisely the poset of partitions of $\{1,2,3,\dots,n\}$.

Eg. $13|254|6$ corresponds to the subspace $\left\{\begin{matrix} x_1 = x_3, \\ x_2 = x_5 = x_4 \end{matrix}\right\}$.

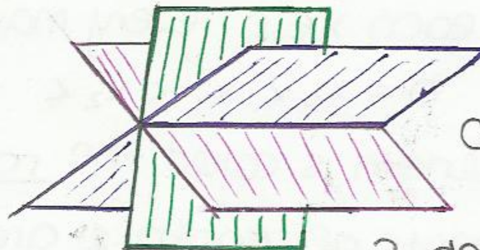
Defn: A p-tuple of atoms $(a_1, a_2, \dots, a_p)$ in a geometric lattice L are <u>independent</u> if $r(a_1 \vee a_2 \vee \cdots \vee a_p) = p$ and <u>dependent</u> if $r(a_1 \vee a_2 \vee \cdots \vee a_p) < p$.

For $L(\mathcal{A})$, a p-tuple of hyperplanes $H_1, \dots, H_p$ is
independent if $\operatorname{codim}(H_1 \cap \cdots \cap H_p) = p$
dependent if $\operatorname{codim}(H_1 \cap \cdots \cap H_p) < p$.



codim = 3

3 independent hyperplanes (general position)



codim = 2

3 dependent hyperplanes

Thm (Orlik–Solomon): Let $\mathcal{A}$ be a (finite) hyperplane arrangement.
The algebra $H^*(M(\mathcal{A}); \mathbb{C})$ has the following structure:

Define E to be the exterior algebra on the basis $\{e_H \mid H \in \mathcal{A}\}$.
Define I in E to be the ideal generated by elements

$$\left\{ \sum_{k=1}^{p} (-1)^{k-1} e_{H_1} e_{H_2} \cdots \widehat{e_{H_k}} \cdots e_{H_p} \;\middle|\; (H_1, H_2, \dots, H_p) \text{ is dependent} \right\}$$

Then $H^*(M(\mathcal{A}); \mathbb{C}) \cong E/I$ as graded algebra.
If $G \curvearrowright \mathcal{A}$, then this is an isomorphism of G-representations.

Rmk: E/I is defined entirely by the data of the lattice $L(\mathcal{A})$.
E and E/I are in fact functors (covariant)

$$\text{Geometric Lattices} \longrightarrow \begin{matrix}\text{graded anticommutative} \\ \mathbb{C}\text{-algebras}\end{matrix}$$

where we requre morphisms $f: L \to L'$ of geometric lattices to satisfy:

$$f(\text{atoms}_L) \subseteq \text{atoms}_{L'} \qquad r_{L'}(fx) \leq r_L(x).$$
$$fx \vee fy = f(x \vee y)$$

<u>Exercise</u>: Show Arnold's presentation for $H^*(PBr_n; \mathbb{Z})$ coincides with the Orlik-Solomon algebra.

<u>Rmk</u> In $H^p(M(A); \mathbb{C})$, $e_{H_1} \cdot e_{H_2} \cdots e_{H_p} = 0$ if $H_1, \ldots, H_p$ are dependent.

Just multiply $\sum (-1)^{k-1} e_{H_1} \cdots \hat{e}_{H_k} \cdots e_{H_p}$ through by e_{H_p}. Since $e_{H_p} \cdot e_{H_p} = 0$, the only surviving term is $\pm e_{H_1} \cdots e_{H_p}$.

<u>Rmk</u> Let $A = E/I$, and let $A_X = \text{span}\langle e_{H_1} \cdots e_{H_p} \mid H_1 \cap \cdots \cap H_p = X \rangle$

Orlik-Solomon prove that $A = \bigoplus_{X \in L(A)} A_X$

By the above remark, $A^p = \bigoplus_{r(X) = p} A_X$.

<u>Proof Outline</u>: Orlik-Solomon Theorem.

1. Prove that the algebra $A = E/I$ associated to a lattice L has Poincaré poly
$$P_A(t) = \sum_{X \in L} \mu(X)(-t)^{r(X)}$$
where $\mu(X) = \mu(0, X)$ is the Mobius function of L

(Proof is involved, though entirely combinatorial / algebraic)

2. For $H \in A$, define forms $\omega_H = \dfrac{d\varphi}{2\pi i \varphi}$ for φ the form defining H

Prove directly (linear algebra & induction) that ω_H satisfy the relations defining I.

Thus $A = E/I \twoheadrightarrow H^*(M(A); \mathbb{C})$.

3. use Brieskorn and induction to show dimensions agree

Fifth
Question For a finite Coxeter gp G, what is $H^*(M(A_G);\mathbb{C})$ as a G-representation?

Thm (Lehrer-Solomon, 1986) Let G be S_n.

There is an isomorphism of $\mathbb{C}S_n$-modules

$$(*) \quad H^p(M(A_{S_n});\mathbb{C}) \cong \bigoplus_{c} \mathrm{Ind}_{Z(c)}^{S_n}(\xi_c) \quad , \ p=0,\dots,n-1$$

- sum runs over reps c of each conj class in S_n with $n-p$ cycles
- ξ_c is an explicitly defined 1-dim rep of the centralizer $Z(c)$ of c in S_n.

In the process of proving this result, Lehrer-Solomon observe the following:

Since $H^p(M(A_G);\mathbb{C}) \cong A^p = \bigoplus_{r(x)=p} A_x$

Let Θ denote the G-orbit of some $x \in L$,

then $A_\Theta := \bigoplus_{x\in\Theta} A_x \cong \mathrm{Ind}_{G_x}^{G} A_x$ by the orbit-stabilizer thm.

In the case that G is a finite Coxeter gp, the stabilizer $G_x = N_G(G_{\cdot x})$, the normalizer in G of the parabolic subgp that fixes x pointwise.

so $H^p(M/A_G);\mathbb{C}) \cong \bigoplus_{\substack{x\in\mathscr{C}\\ r(x)=p}} \mathrm{Ind}_{N_W(G_{\cdot x})}^{G} A_x$

where $\mathscr{C}$ is a set of reps of orbits of G.

Note: In 2010 Church-Farb used this description to show the sequences $\{H^p(M(A_{G_n});\mathbb{C})\}_n$ are uniformly rep stable when G_n is S_n or B_n.

I have observed that they are "Finitely-generated $FI_W\#$-modules" in type A_n, B/C_n and D_n.

Lehrer-Solomon conjectured that analogous decomps to (*) hold for all finite reflection gps.

As of this year (2013), progress has been made on this conjecture by Bishop-Douglass-Pfeiffer-Roehrle.