Polynomial Invariants of Finite Reflection Gps.

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Farb & Friends Geometry Seminar

Review: Coxeter Gps.

Def^n A Coxeter Gp G is a gp with a presentation of the following form:

$$G = \langle s \in S \mid s^2, (st)^{m_{s,t}} \rangle$$

generators are involutions

For $s \neq t$

$m_{s,t} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$

$m_{s,t} = 2 \iff s$ and t commute

$m_{ss} = 1$.

(we will assume G finite)

The pair (G, S) is called a Coxeter system.

Motivating Examples:

1. The Symmetric Gp S_n (Type A_{n-1})

$$S_n = \langle s_1, \ldots, s_{n-1} \mid s_i^2, \ (s_i s_j)^2 \text{ if } |i-j|>1, \ (s_i s_j)^3 \text{ if } |i-j|=1 \rangle$$

$s_i = (i \ i+1)$

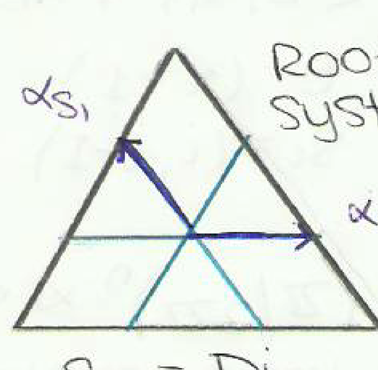
called the braid relations.

Rank: $n-1$

Degrees: $2, 3, 4, \ldots, n$

Eg. $S_3 = \langle s_1, s_2 \mid s_1^2, s_2^2, (s_1 s_2)^3 \rangle$

s_1 = reflection along α_{s_1}
(ie, in hyperplane perp to α_{s_1})
s_2 = reflection along α_{s_2}
$s_1 s_2$ = rotation by $2\pi/3$.



$S_3 = Di_3$

② Dihedral Gp Di_m (acting on m-gon).

$$Di_m = \langle s, t \mid s^2, t^2, (st)^m \rangle$$

Rank 2

Degrees : 2, m.

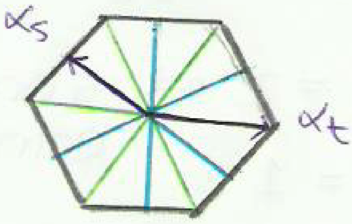
s = reflection along α_s

t = reflection along α_t

st = rotation by $2\pi/m$.

Eg.

$$Di_5 = \langle s, t \mid s^2, t^2, (st)^5 \rangle$$



$Di_6 \cong G_2$

$$Di_6 = \langle s, t \mid s^2, t^2, (st)^6 \rangle$$

Note that the nature of the root system differs in odd and even degree m.

For m even, there are roots of two different lengths.

③ Hyperoctahedral Gp (Gp of signed permutation matrices).

Type B_n / C_n.

$$B_n = \langle t, s_1, \dots, s_{n-1} \mid \begin{array}{l} s_i^2,\ t^2,\ (s_1 t)^4,\ (s_i t)^2 \text{ for } i > 1. \\ (s_i s_j)^3 \text{ if } |i-j| = 1 \\ (s_i s_j)^2 \text{ if } |i-j| < 1 \end{array} \rangle$$

$t = (1 \ {-1})$

$s_i = (i \ \ i+1)$

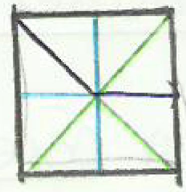
Rank n

Degrees 2, 4, 6, …, n.

$B_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$.

B_n is the symmetry gp of an n-cube.

<u>Eg</u> $B_2 = \langle t, s \mid s^2, t^2, (st)^4 \rangle$



$B_2 \cong Di_4$

The Theory of Coxeter Gps

<u>Defn</u> $|S| = n$ is the <u>rank</u> of a Coxeter gp G

Given a Coxeter system (G, S), there is a procedure for associating a <u>root system</u> to (G, S), which gives G the structure of reflection gp on $\mathbb{R}^n$, $n = |S|$.

- Take the $\mathbb{R}$-vector space with basis $\{\alpha_s \mid s \in S\}$. These are called the simple roots.
- Define action of G so that $s \in S$ acts by

$$\alpha_t \cdot s = \alpha_t + 2\cos\left(\frac{\pi}{m_{s,t}}\right) \alpha_s$$

s acts by a "reflection along α_s" in the sense that α_s is an eigenvector of s with eigenvalue -1, and has a direct complement on which s acts trivially.

(Check: Its trace is $n-2$, and its minimal polynomial is determined by $s^2 = $ identity)

<u>Defn</u> The orbit $\Phi = \{\alpha_s \cdot G\}$ is the <u>root system</u> associated to (G, S). Its elements are <u>roots</u>.

The Coxeter Element

Defn A Coxeter elt in a Coxeter system (G, S) is a product of all the generators $\prod_{s \in S} s$.

Fact All Coxeter elts are conjugate.

Defn The order of a Coxeter elt is the Coxeter number of (G, S), often denoted h.

Fact $h = |\Phi| / n$. Φ = root system, n = rank.

Eg S_n – Coxeter elts are n-cycles

Reflection Gp

Defn A real reflection gp is a discrete gp G with an action on $\mathbb{R}^n$ generated by reflections.

Fact: The finite real reflection gps are precisely the finite Coxeter gps.

Polynomial Invariants of Reflection Gps.

Setup: Let V be a finite-dim vector space over a field K of characteristic 0.

Let $n = \dim(V)$.

Defn A reflection in $GL(V)$ is a linear map with fixed set a hyperplane in V.

Let $G \leq GL(U)$ be a finite gp generated by reflections.

Denote $S = S(U^*)$ the symmetric algebra on the dual U^* of U.

Note $S \cong k[x_1, \dots, x_n]$ polynomials in the coordinate fxns x_i of U.

The action $G \curvearrowright U$ induces $G \curvearrowright S$ by:

$$g \cdot f \longmapsto \{ u \longmapsto f(g^{-1}(u)) \}$$

Note the action preserves the grading.

<u>Defn</u> $f \in S$ is G-invariant if $g \cdot f = f \;\forall g \in G$.

ie, f is G-invariant if f is constant on orbits of G.

Denote $R := S^G$ the (homogeneous) subalg of G-invariants.

Note: If $G = S_n \curvearrowright \mathbb{R}^n$ by permutation matrices, then R is the algebra of symmetric poly's.

$S_n \curvearrowright k[x_1, \dots, x_n]$ by permuting indices.

Our goal: Generalize theory of symmetric polynomials to other reflection gps.

<u>Fact</u> "The Fundamental Thm of Symmetric Poly's"

The symmetric polynomials $k[x_1, \dots, x_n]^{S_n}$ are generated by the n elts $e_1, e_2, \dots, e_n$ (and 1)

$e_1 := x_1 + x_2 + \cdots + x_n$

$e_2 := x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n$

$\cdots$

$e_n = x_1x_2 \cdots x_n$.

These are the <u>elementary symmetric poly's</u>. They are algebraically independent, ie, every symmetric poly is given by a unique poly in the e_i's.

This phenomenon generalizes to other reflection gps:

<u>Thm</u> (Chevalley, 1950's)

R is generated as a k-algebra by $n = \dim(V)$ homogeneous, algebraically independent elts of positive degree (and 1).

ie, $R \cong$ a free polynomial alg.

<u>Fact</u> A set of homogeneous generators are called <u>basic invariants</u> of G. These generators are not uniquely determined, but their degrees are. These are called the <u>degrees</u> of G.

Chevalley's thm has a converse, proved by Shephard and Todd.

<u>Thm</u> (Shephard-Todd, 1950s) If the invariants $R = S^G$ of a finite gp $G \leq GL(V)$ has algebraically independent generators, then G is generated by its reflections.

<u>Outline of Proof of Chevalley's Thm</u>
(Source: Humphreys, <u>Reflection gps & Coxeter gps</u>)

<u>Step ①</u> "R is not too small"

<u>Propn</u> The field of fractions $\mathrm{Frac}(R)$ is equal to the subfield of G-invariants of $\mathrm{Frac}(S)$.

$$\begin{array}{cl} \left.\begin{array}{c} K(x_1,\dots,x_n)\\ | \\ K(x_1,\dots,x_n)^G = \mathrm{Frac}(R)\end{array}\right\} & \text{finite extension Galois gp } G. \\ \left.\begin{array}{c} | \\ K\end{array}\right\} & \text{transcendence deg } n.\end{array}$$

In particular, R has transcendence deg n over K.

Proof

clearly $\mathrm{Frac}(R) \hookrightarrow k(x_1, \ldots, x_n)^G$

since given any G-invariants f, g, the quotient f/g is G-invariant.

Conversely, given $\frac{p}{q} \in k(x_1, \ldots, x_n)^G$,

$$\frac{p}{q} = \frac{\prod_{g \neq 1} (g \cdot p)}{\prod_{g \neq 1} (g \cdot p)} \frac{p}{q}$$

- numerator $\prod_{g \in G} pg$ is G-invariant, and the quotient is G-invariant by assumption, so the denominator must be G-invariant.

Thus $\frac{p}{q}$ can be realized as a quotient of elts of R.

$\mathrm{Frac}(R) = k(x_1, \ldots, x_n)^G$ $\square$.

Step ② "R is not too big".

Prop^n R is finitely generated as a k-algebra.

Specifically, let $R^+ \subseteq R$ be all poly's with constant term 0, and let $I = SR^+$ be the ideal they generate.

Then I has some finite homogeneous generating set $f_1, \ldots, f_r \in R^+$, and this set generates R as a k-algebra (along with 1).

Note: That SR^+ is finitely generated as an ideal follows from Hilbert's basis thm.

Proof

Requires char(k) to be coprime to $|G|$.

Main tool Projection map

$$S \longrightarrow R$$
$$f \longmapsto f^\# := \frac{1}{|G|} \sum_{g \in G} g \cdot f$$

Exercise: $p \in S$, $q \in R$ then $(pq)^\# = p^\# q$.

We want to show that any $f \in R$ is in the k-algebra generated by $f_1, \dots, f_r$.
Proceed by induction on degree (automatic in degree 0).

For $f \in R$, write $f = \sum_{i=1}^{r} s_i f_i$ $s_i \in S$ homogeneous.

then
$$\begin{aligned} f^\# = f &= (s_1 f_1 + \cdots + s_r f_r)^\# \\ &= s_1^\# f_1 + \cdots + s_r^\# f_r \end{aligned}$$

homogeneous elts of R of degree strictly less than f; Proof follows by induction. □

Step 3 Take a minimal generating set $f_1, \dots, f_r$ as above. These generators are algebraically independent.

<u>Note</u> Since R has transcendence deg n over k, it follows that $r = n$.

<u>Proof</u> This proof is more combinatorially involved. The idea is to assume an algebraic relation between the poly's $f_1, \dots, f_r$, and show that this contradicts the assumed minimality of this generating set.

Step ③ is the only point that we use the fact that G is a reflection gp, as follows:

Suppose s is a reflection in G, which fixes the hyperplane defined as the zero set of a linear poly ℓ.

Then if $g \in S$, $sg - g$ vanishes on this hyperplane and so factors through ℓ.

That $\frac{sg-g}{\ell}$ is lower degree than $sg - g$ can be used in an induction argument.

For details, see Humphreys. □.

<u>Surprising Combinatorial Properties of Degrees</u>

Let G be a finite reflection gp with degrees $d_1, \dots, d_n$.

<u>Fact</u> $|G| = \prod_{i=1}^{n} d_i$. <u>Fact</u> $\#$ reflections in G $= \sum d_i - n$.

Fact If W is a Weyl gp, the degrees determine the # of roots of each height.

Fact If action of G is irreducible,

- $Z(G)$ is cyclic, and $|Z(G)| = GCD(d_i)$
- highest degree d_n is the Coxeter # h.

Fact If g is a Coxeter elt, it has order h, so its eigenvalues are ζ^{m_i} for some hth root of unity ζ and integers m_i (reduced mod h)

Then $m_i = d_i - 1$. These are called the exponents of G.

Fact The Poincaré polynomial $W(t)$ associated to (G, S) is defined by

$$W(t) = \sum a_m t^m = \sum_{g \in G} t^{\ell(g)}$$

where $\ell(g)$ is length of elt g wrt generating set S.

a_m: # elts in G of length m.

Eg. In S_3, $W(t) = 1 + 2t + 2t^2 + t^3$

Thm $W(t) = \prod_{i=1}^{n} \dfrac{t^{d_i} - 1}{t - 1}$

We will see that $W(t)$ is in fact the Poincaré polynomial of a flag variety associated to G.

Coinvariant Algebras

Defn $\mathscr{C} = S/SR_+$ is a coinvariant algebra, a graded algebra over K with an induced action of G

Note: $\mathscr{C}$ is not S mod the action of G.

Fact $\dim_K(\mathscr{C}) = |G|$
and the dimensions of the graded pieces are given by the generating function $W(t)$, the Poincaré poly of G.

Fact As a G-representation,
$\mathscr{C} \cong KG$, the regular rep.

Remark: Coinvariant algebras in algebraic combinatorics

Taking a finite reflection gp $G \curvearrowright V$ as above, there is an induced diagonal action of G on $V^{\oplus r}$ (the generators no longer act by reflections).

We can again consider the G-invariants of the poly algebra $S(V^{\oplus r})^*$, and construct the coinvariant algebra.

The theory of finite reflection gps no longer applies, and there are many open questions about the combinatorics of these coinvariant algebras, called <u>diagonal</u> coinvariant algebras (including, in most cases, the dimensions of the graded pieces).

Church-Ellenberg-Farb studied these as part of their "representation stability" program, and shown that the graded pieces of the diagonal coinvariant algebras associated to the symmetric gp are co-FI-modules, and hence the pattern of decomposition into irreducible S_n-reps, and their S_n-rep characters, satisfy some strong constraints and "stabilize" in some precise sense. I have shown the same for the diagonal coinvariant algebras associated to the Weyl gps in type B/C and D.

<u>Connections of Coinvariant Algebras to Generalized Flag Varieties</u>

Let H be a complex semisimple Lie Gp

Let B be a Borel subgp
(a maximal connected soluable subgp)

Let K be a maximal cpt subgp
Let $T \leq K$ be a maximal torus.

Then $K/T \overset{\text{homeomorphism}}{\cong} H/B$ is called a <u>generalized flag variety</u>

The prototypical example is Type A_{n-1}

$H = SL_n\mathbb{C}$
$B =$ upper triangulars
$U =$ unitary gp
$T =$ diagonal matrices.

Then $SL_n\mathbb{C}/B$ is the space of complete flags in $\mathbb{C}^n$,

points are sequences of subspaces
$\{0 < U_1 < \cdots < U_{n-1} < \mathbb{C}^n\}$
$\dim(U_i) = i$.

an $\frac{n(n-1)}{2}$-dim manifold.

<u>Defn</u> The Weyl gp associated to H is

$$G = N(T)/T, \quad N(T) = \text{normalizer of } T \text{ in } K.$$

In type A_{n-1}, H is S_n.

Thm (Borel)

Let $X(T)$ be the group of characters of T

ie, $X(T) = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{C}^*)$

For type A_{n-1}, $X(T) \cong \mathbb{Z}^{n-1}$

$$\mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{C}^*) \longleftarrow \mathbb{Z}^n$$

$$\left\{ \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{bmatrix} \longmapsto t_1^{m_1} \cdots t_n^{m_n} \right\} \longleftarrow (m_1, \ldots, m_n)$$

Since $t_1 \cdots t_n = 1$, this map has kernel the diagonal $(m, m, \ldots, m) \subseteq \mathbb{Z}^n$.

The Weyl gp $S_n \curvearrowright \mathbb{Z}^{n-1} = \mathbb{Z}^n / \text{diagonal}$.

Let $\mathcal{C}$ be the coinvariant algebra associated to $G \curvearrowright U = \underbrace{X(T) \otimes_{\mathbb{Z}} \mathbb{C}}_{\text{complexified character gp.}}$

Then there is an isomorphism of graded $\mathbb{C}G$-modules:

$$c: \mathcal{C} \xrightarrow{\cong} H^*(H/B, \mathbb{C})$$
$$\| $$
$$K/T$$

$$\text{character } \chi \longmapsto \text{First Chern class associated to Line bundle } L_\chi$$

where $L_\chi := K \times \mathbb{C} \Big/ (k,z) \sim (kt^{-1}, \chi(t)z)$

For more details on this map (and its relation to the Schubert cell description of $H^*(H/B)$)
see Hiller, <u>the Geometry of Coxeter Gps.</u>