Michigan Representation Stability Week 2018
University of Michigan, Ann Arbor • 13–17 August 2018

An introduction to FI–modules and their generalizations

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These notes and exercises accompany a 3-part minicourse on FI–modules and their generalizations. More advanced exercises are marked with an asterisk. These notes assume the following prerequisites:

- basic theory of modules over a ring, including Noetherian rings, and tensor products and multilinear algebra,
- basic category theory, including the definition of categories, functors, natural transformations, and adjoint functors,
- basic representation theory of finite groups and character theory, including the structure of induced representations
- basic representation theory of the symmetric groups, including the classification of irreducible representations,
- basic homological algebra, including the definition of projective modules, basic properties of homology groups of spaces and groups, and the structure of a spectral sequence.

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1 Some goals of representation stability

A major goal of the field of representation stability is to develop tools to understand the algebraic structures that govern certain naturally-arising sequences of group representations. For example, the symmetric groups S_n act on the (co)homology groups of the following families of groups and spaces, and each sequence of (co)homology groups has a common underlying structure: it is an FI–*module*.

- Hyperplane complements associated to certain reflection groups containing S_n,
- Configuration spaces of n ordered points in a manifold M or graph G,
- Congruence subgroups $\mathrm{GL}_n(\mathbb{Z}, p)$ of $\mathrm{GL}_n(\mathbb{Z})$,
- Complete flag varieties associated to $\mathrm{GL}_n(\mathbb{Z})$,
- The pure mapping class groups of a surface with n marked points.

In general, the theory of FI–modules is designed to address the following framework. Consider a sequence of S_n–representations

$$V_0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow \cdots ,$$

usually over $\mathbb{Q}$ or $\mathbb{Z}$, such that the maps $V_n \to V_{n+1}$ are S_n–equivariant. The objectives of the program typically involve the following.

Goal (a). Characterize the algebraic structure of the sequence of representations $\{V_n\}$, often by realizing the sequence as an FI–module or an object in a suitably defined functor category;

Goal (b). Prove finiteness results for the module $\{V_n\}$, such as bounds on the degree of generators and relations, or vanishing of associated functor homology groups;

Goal (c). Deduce structural results on $\{V_n\}$ from these finiteness properties. This may mean

- (For semi-simple representations) establish constraints on the irreducible constituents of V_n or on the characters $\{\chi_{V_n}\}$ for n large, such as *multiplicity stability* or *polynomial characters*.
- Compute an explicit formula for V_n as an S_n–representation in terms of the earlier terms $V_0 \to V_1 \to \ldots \to V_{n-1}$ for all n sufficiently large.
- Bound the growth rate in n of $\text{rank}(V_n)$. For example, it may be polynomial of bounded degree.

Results of this form are called *representation stability* for the sequence $\{V_n\}$.

The literature on these results involves the work of more authors than can practically be listed here; see for example the survey [Wi3] for an extensive list of references. The program as it is presented in these notes builds largely on the foundational work of Church, Ellenberg, Farb, and Nagpal [CF, CEF1, CEFN, CEF2, CE], Sam and Snowden [Sn, SS2, SS1], Putman [Pu], and Putman and Sam [PS].

2 The category FI and FI–modules

In this lecture we will develop the algebraic foundations of the theory of FI–modules. This lecture draws heavily on Church–Ellenberg–Farb [CEF1].

2.1 The definition of an FI–modules

Definition I. (The category FI.) Let FI denote the category whose objects are finite sets (including $\emptyset$) and whose morphisms are all injective maps.

Church–Ellenberg–Farb [CEF1] used the notation FI for this category as an acronym for Finite sets and Injective maps. The category has appeared in other contexts in algebraic topology, algebraic geometry, and computer science under various names, such as $\mathbb{I}$, Π, F_{inj}, Set_{fm}, *Inj*, etc.

Notation II. For a positive integer n, we write

$$[n] := \{1, 2, \ldots, n\},$$

and we write $[0]$ to denote the empty set. We let $\iota_{m,n}$ denote the canonical inclusion

$$\iota_{m,n} : [m] \hookrightarrow [n].$$

Exercise 1. (An equivalence of categories). A functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* if it satisfies the following:

- F is *full*. This means that for any pair of objects $c_1, c_2 \in \mathcal{C}$, the map

$$\text{Hom}_{\mathcal{C}}(c_1, c_2) \longrightarrow \text{Hom}_{\mathcal{D}}(F(c_1), F(c_2))$$

induced by F is surjective.

- F is *faithful*. This means that for any pair of objects $c_1, c_2 \in \mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(c_1, c_2) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(c_1), F(c_2))$$

  induced by F is injective.
- F is *essentially surjective*. This means every object $d \in D$ is isomorphic to an object of the form $F(c)$, for $c \in \mathcal{C}$.

(a) Consider the full subcategory of FI of finite sets of the form $[n]$, $n \in \mathbb{Z}_{\geq 0}$. Show that the inclusion of this subcategory into FI is an equivalence of categories.

(b) A *skeleton* of a category $\mathcal{D}$ is an equivalent category $\mathcal{C}$ in which no two distinct objects are isomorphic. Conclude that the full subcategory of FI of finite sets $[n]$ is a skeleton of FI. By abuse of terminology some authors also refer to this skeleton category as FI.

Exercise 2. Consider the skeleton subcategory of FI of sets $[n]$.

(a) Verify that for $n \in \mathbb{Z}_{\geq 0}$, the endomorphisms of the finite set $[n]$ are the symmetric groups

$$\mathrm{End}_{\mathsf{FI}}([n]) \cong S_n.$$

Here we use the convention that $S_0 = S_1$ is the trivial group.

(b) Show that the morphisms in this category are generated by the endomorphisms S_n and the inclusions $\iota_{n,n+1}$.

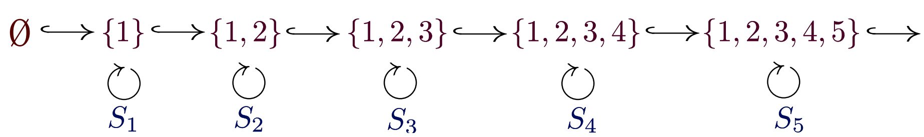


Figure 1: A skeleton of FI

Exercise 3.

(a) Show that the endomorphisms $\mathrm{End}_{\mathsf{FI}}([n]) \cong S_n$ act on the set of morphisms $\mathrm{Hom}_{\mathsf{FI}}([m],[n])$ on the left by postcomposition, that is,

$$\begin{aligned} \sigma : \mathrm{Hom}_{\mathsf{FI}}([m],[n]) &\longmapsto \mathrm{Hom}_{\mathsf{FI}}([m],[n]) \\ \alpha &\longmapsto \sigma \circ \alpha \qquad\qquad \text{for all } \sigma : [n] \to [n] \end{aligned}$$

(b) Show that this action is transitive.

(c) Show that the stabilizer of the inclusions $\iota_{m,n}$

$$\{ \sigma \in S_n \mid \sigma \circ \iota_{m,n} = \iota_{m,n} \}$$

is isomorphic to S_{n-m}.

(d) Conclude that, as an S_n–set,

$$\mathrm{Hom}_{\mathsf{FI}}([m],[n]) \cong S_n / S_{n-m}.$$

Exercise 4.

(a) Show that the endomorphisms $\mathrm{End}_{\mathsf{FI}}([m]) \cong S_m$ act on the set of morphisms $\mathrm{Hom}_{\mathsf{FI}}([m],[n])$ on the right by precomposition, that is,

$$\sigma : \mathrm{Hom}_{\mathsf{FI}}([m],[n]) \longmapsto \mathrm{Hom}_{\mathsf{FI}}([m],[n])$$
$$\alpha \longmapsto \alpha \circ \sigma \qquad \text{for all } \sigma : [m] \to [m]$$

(b) Determine whether this action is transitive.

Definition III. (FI–modules.) Let R be a commutative ring. Define an FI*–module* V over R to be a (covariant) functor from FI to the category of R–modules.

All rings are assumed to have unit. Frequently the ring R will be $\mathbb{Z}$ or $\mathbb{Q}$.

Notation IV. Let R be a commutative ring, and V an FI–module over R. For a finite set S, we write V_S to denote the R–module $V(S)$, and for $n \in \mathbb{Z}_{\geq 0}$ we sometimes write V_n in the case $S = [n]$. For an FI morphism α we often write α_* or (if there is no confusion) simply α to denote the R–module map $V(\alpha)$.

Exercise 5. Let V be an FI–module over a ring R.

(a) Show that for each $n \in \mathbb{Z}_{\geq 0}$ the action of $\mathrm{End}_{\mathsf{FI}}([n]) \cong S_n$ on the R–module V_n gives V_n the structure of an S_n–representation. Equivalently, V_n is an $R[S_n]$–module.

(b) The inclusion $\iota_{m,n} : [m] \to [n]$ defines an embedding $S_m \hookrightarrow S_n$. Show that the map $(\iota_{m,n})_* : V_m \to V_n$ must be S_m–equivariant with respect to the action of S_m on V_m, and the action of the subgroup $S_m \subseteq S_n$ on V_n.

For many purposes, it is expedient to simply consider the restriction of an FI–module V to the skeleton of FI of sets $[n]$ (Exercise 1). Every object in the image of V is determined up to isomorphism. Under this restriction, we may view V as a sequence of $R[S_n]$–modules V_n along with equivariant maps $V_m \to V_n$. By abuse of terminology, these restricted functors are sometimes also called FI–modules in the literature.

Exercise 6. Let V be a functor from the category of finite sets $[n]$ and injective maps to the category of R–modules. Show that the structure of V is completely determined by the sequence of S_n–representations V_n and the maps $(\iota_{n,n+1})_* : V_n \to V_{n+1}$, as in Figure 2. See Exercise 2(b).

In these notes we will generally be agnostic as to whether an FI–module is a functor from FI or only from the full subcategory of finite sets $[n]$. In practice the question of which convention to adopt is often a matter of notation convenience. It may be simpler to view FI–modules just as sequences $\{V_n\}$ of S_n–representations with additional maps. On the other hand, this formulation can be more notationally cumbersome if we wish, for example, to use operations induced by disjoint unions of sets, as we then need to make choices of identifications $[m] \sqcup [n] \cong [m+n]$.

2.2 Examples and non-examples of FI–modules

Exercise 7. (Examples of FI–module.) Show that each of the following sequences has the structure of an FI–module over $\mathbb{Z}$.

(a) $V_n = \mathbb{Z}$ the trivial S_n–representations, all maps are isomorphisms

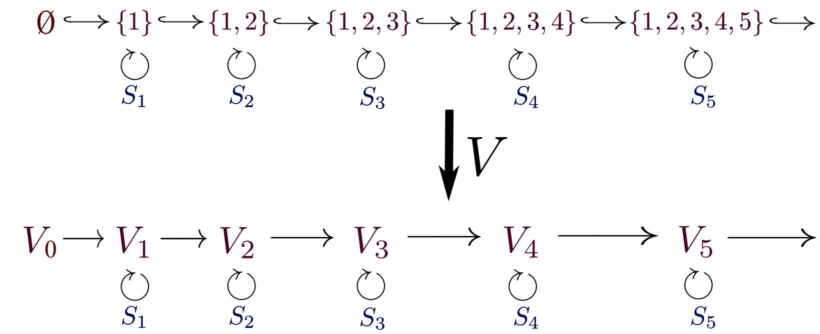


Figure 2: The structure of an FI–module

(b) $V_n = \mathbb{Z}^n$ the canonical permutation representations, maps $V_n \to V_{n+1}$ the natural inclusions

(c) $V_n = \bigwedge^k(\mathbb{Z}^n)$, maps $V_n \to V_{n+1}$ the natural inclusions

(d) V_n any sequence of S_n–representations, all maps $V_m \to V_n$ with $n > m$ are zero

(e) $V_n = \mathbb{Z}[x_1, \ldots, x_n]$ the polynomial algebra, maps $V_n \to V_{n+1}$ the natural inclusions

(f) $V_n = \mathbb{Z}[x_1, \ldots, x_n]_{(k)}$ the homogeneous polynomials in $x_1, \ldots, x_n$ of fixed degree k, maps $V_n \to V_{n+1}$ the natural inclusions

(g) $V_n = \mathbb{Z}[S_n]$ with action of S_n by conjugation, maps $V_n \to V_{n+1}$ the natural inclusions

Exercise 8. Let V be any FI–module. Show that the following constructions yield new FI–modules.

(a) **(torsion FI–modules)** For $N \in \mathbb{N}$, define $V^{\leq N}$ such that

$$V_n^{\leq N} = \begin{cases} V_n, & n \leq N \\ 0, & n > N. \end{cases}$$

Morphisms with codomain $\leq N$ agree with those of V, and morphisms with codomain $> N$ are zero.

(b) **(truncated FI–modules)** For $N \in \mathbb{N}$, define $V^{\geq N}$ such that

$$V_n^{\geq N} = \begin{cases} 0, & n < N \\ V_n, & n \geq N. \end{cases}$$

Morphisms with domain $\geq N$ agree with those of V, and morphisms with domain $< N$ are zero.

(c) Postcomposing V with any functor R–Mod $\to R$–Mod, such as $\bigwedge^k$, $\bigwedge^*$, Sym^k, Sym^*, $\otimes^k$, or $\otimes^*$ (for $k \in \mathbb{Z}_{\geq 0}$).

We have seen that an FI–module V is defined by a sequence of S_n–representations V_n along with S_n–equivariant maps $V_n \to V_{n+1}$. The converse, however, is not quite true. The following exercise determines when such a sequence arises from an FI–module.

Exercise 9. (The FI–module criterion.) Suppose that $\{W_n\}$ is a sequence of S_n–representations with S_n–equivariant maps $\phi_n : W_n \to W_{n+1}$. Let $G \cong S_{n-m}$ denote the stabilizer of $\iota_{m,n}$ under the action of S_n by postcomposition (Exercise 3). Show that $\{W_n\}$ can be promoted to an FI–module with $(\iota_{n,n+1})_* = \phi_n$ if and only if

$$\text{for all } m < n, \qquad \sigma \cdot v = v \qquad \text{for all } \sigma \in G \text{ and } v \in \text{im}((\iota_{m,n})_*).$$

The FI–module criterion of Exercise 9 implies that the following naturally-arising equivariant sequences of S_n–representations do not in fact have FI–module structures.

Exercise 10. (Non-examples of FI–module.) Show that the following sequences do **not** have the structure of an FI–module.

(a) $V_n = \mathbb{Z}$ the alternating representations, all maps are isomorphisms

(b) $V_n = \mathbb{Z}[S_n]$ the left regular representations (that is, with action of S_n by left multiplication), maps $V_n \to V_{n+1}$ the natural inclusions.

2.3 FI–submodules and maps of FI–modules

Definition V. Given an FI–module V, a *submodule* U of V is a sequence of S_n–subrepresentations $U_n \subseteq V_n$ that is closed under the action of the FI–morphism.

Definition VI. (Maps of FI–modules.) Fix a commutative ring R. A *map of* FI*–modules* $V \to W$ over R is a natural transformation of functors. Concretely, a map $F : V \to W$ of FI–modules is a sequence of maps

$$F_n : V_n \to W_n \qquad \text{for all } n \in \mathbb{N}$$

making the following diagrams commute for every n and every FI morphism $\alpha : [m] \to [n]$.

$$\begin{array}{ccc} V_m & \xrightarrow{F_m} & W_m \\ {\scriptstyle \alpha_*}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ V_n & \xrightarrow{F_n} & W_n \end{array}$$

Exercise 11. Let $F : V \to W$ be a map of FI–modules. Show that, for each n, the map $F_n : V_n \to W_n$ must be S_n–equivariant.

Exercise 12. (The map of FI–modules criterion.) Let V and W be FI–modules over a ring R. Let $F_n : V_n \to W_n$ be a sequence of S_n–equivariant maps. Show that, to verify that F is a map of FI–modules, it suffices to check that the diagrams

$$\begin{array}{ccc} V_n & \xrightarrow{F_n} & W_n \\ {\scriptstyle \iota_{n,n+1}}\downarrow & & \downarrow{\scriptstyle \iota_{n,n+1}} \\ V_{n+1} & \xrightarrow[F_{n+1}]{} & W_{n+1} \end{array}$$

commute for each n. See Exercise 2(b).

Just as with modules over a ring, FI–modules have natural notions of images, kernels, cokernels, quotients, etc ..., all defined pointwise.

Exercise 13. (Images of FI–module maps.) Fix a commutative ring R, and let $F : V \to W$ be a map of FI–modules over R. Show that the sequence of $R[S_n]$–submodules

$$F_n(V_n) \subseteq W_n$$

forms an FI–submodule of W. This submodule is called the *image of* F and written $F(V)$ or $\text{im}(F)$.

Exercise 14. (Kernels of FI–module maps.) Fix a commutative ring R, and let $F : V \to W$ be a map of FI–modules over R. Show that the sequence of $R[S_n]$–submodules

$$\ker(F_n) \subseteq V_n$$

forms an FI–submodule of V. This submodule is called the *kernel of* F and written $\ker(F)$.

Exercise 15. (FI–module quotients and cokernels.) Let V be an FI–module over a ring R, and let $U \subseteq V$ be a submodule.

(a) Show that the sequence of S_n–representations V_n/U_n has the structure of FI–module. We call this FI–module the *quotient* of V by U, and denote it by V/U.

(b) Show that the sequence of maps

$$Q_n : V_n \twoheadrightarrow V_n/U_n$$

define a map of FI–modules. This is called the *quotient map* of V by U.

(c) Let $F : V \to W$ be a map of FI–modules over R. As a special case, conclude that cokernels

$$\text{coker}(F_n) := W_n/F_n(V_n)$$

form an FI–module. We call this FI–module the *cokernel* of the map F and write $\text{coker}(F)$.

Consider the following examples and non-examples of maps of FI–modules.

Exercise 16. (Examples and non-examples of maps of FI–modules).)

(a) Let $\mathbb{Q}^n$ denote the canonical permutation representation of the symmetric group S_n, and let $\mathbb{Q}$ denote the 1–dimensional trivial S_n–representation.

  (i) Show that, for fixed n, the trivial representation $\mathbb{Q}$ is both a subrepresentation and quotient of the S_n–representation $\mathbb{Q}^n$.

  (ii) Let U denote the FI–module with the trivial S_n–representation $U_n = \mathbb{Q}$ in each degree $n \geq 1$, and all maps isomorphisms. Let V denote the FI–module with the permutation representation $V_n = \mathbb{Q}^n$ in each degree $n \geq 1$ and all maps the natural injections. Show that U is a quotient FI–module of V, but that U is **not** an FI–submodule of V.

(b) Again let U be the FI–module with $U_n = \mathbb{Q}$ in each degree $n \geq 1$, and all maps isomorphisms. Recall the definition of truncated and torsion FI–modules from Exercise 7.

  (i) Show that $U^{\geq 11}$ is a submodule of U, but not a quotient of U.

(ii) Show that $U^{\leq 10}$ is a quotient of U, but not a submodule of U.
(iii) What is the kernel of the quotient map $U \to U^{\leq 10}$?

Exercise 17. (The category of FI–modules over R.) Fix a commutative ring R. Show that FI–modules over R and maps of these FI–modules form a category. We call this the *category of FI–modules over R.*

Exercise 18. Show that the category of FI–modules is not semisimple.

2.4 Sums and tensor products of FI–modules

Exercise 19. (Direct sums of FI–modules.) Let V and W be FI–modules over a ring R. Recall that the direct sums $V_n \oplus W_n$ are S_n–representations with the diagonal action

$$\begin{aligned} \sigma : V_n \oplus W_n &\longrightarrow V_n \oplus W_n \qquad\qquad (\sigma \in S_n) \\ (v, w) &\longmapsto (\sigma \cdot v, \sigma \cdot w). \end{aligned}$$

Show that the sequence of representations $V_n \oplus W_n$ has the structure of an FI–module with a diagonal action of the FI–morphisms. This FI–module is called the *direct sum* of V and W and written $V \oplus W$.

Exercise 20. (Pointwise tensor products of FI–modules.) Let V and W be FI–modules over a ring R. Recall that the tensor products $V_n \otimes_R W_n$ are S_n–representations with the diagonal action

$$\begin{aligned} \sigma : V_n \otimes_R W_n &\longrightarrow V_n \otimes_R W_n \qquad\qquad (\sigma \in S_n) \\ v \otimes w &\longmapsto (\sigma \cdot v) \otimes (\sigma \cdot w). \end{aligned}$$

Show that the sequence of representations $V_n \otimes W_n$ has the structure of an FI–module with a diagonal action of the FI–morphisms. This FI–module is called the *(pointwise) tensor product* of V and W and written $V \otimes_R W$.

3 Generation degree of FI–modules

3.1 Generation of FI–modules

Definition VII. (Generation of FI–modules). An FI–module $V = \{V_n\}$ is *generated* by a set $S \subseteq \coprod_{n \geq 0} V_n$ if V is the smallest FI–submodule containing S.

Equivalently, V is generated by the set S if for each $n \in \mathbb{Z}_{\geq 0}$ the R–module V_n is generated by the images of S under the action of the FI morphisms.

Notation VIII. For V an FI–modules and $v \in V_d$, we write $|v| = d$ and call $|v|$ the FI–*degree* of v.

Definition IX. (Finite generation and generation degree of FI–modules). An FI–module $V = \{V_n\}$ is *generated in degree* $\leq d$ if V is generated by the set $\coprod_{0 \leq n \leq d} V_n$. If V is generated in degree d for some $d < \infty$, then we say that V has *finite generation degree*. If V is generated by some finite set, we say that V is *finitely generated*.

Note that finitely generated FI–modules necessarily have finite generation degree, though the converse need not hold if we allow the R–modules V_n to be infinitely generated.

Example X. (An example of FI–module finitely generated in degree 2.) For example, let V be the FI–module over R with V_n the R–module of homogeneous degree–2 polynomials in the variables $x_1, \ldots, x_n$ with the diagonal action of S_n permuting the indices, and maps $(\iota_{n,n+1})_* : V_n \hookrightarrow V_{n+1}$ the inclusions. Then V is finitely generated in degree ≤ 2 by the monomials $x_1^2 \in V_1$ and $x_1 x_2 \in V_2$, as in Figure 3. Observe that the S_n–orbits of x_1^2 and x_1x_2 form an R–basis for V_n for each $n \geq 1$.

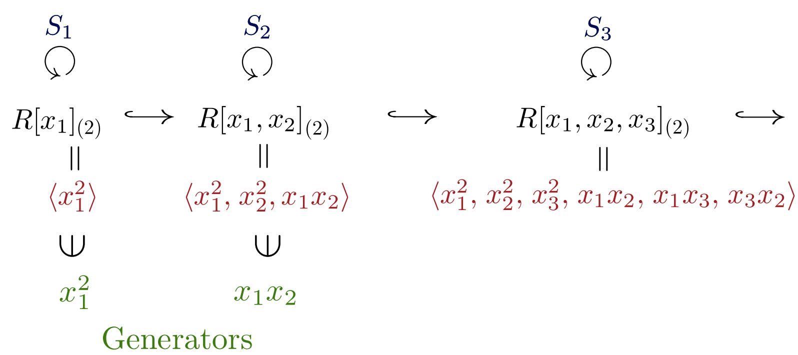


Figure 3: Generators for the FI–module of homogeneous degree 2 polynomials over R

Exercise 21. Determine whether each of the FI–modules of Exercise 7 is generated in finite degree, and which is finitely generated. For those that are generated in finite degree, find a bound on their degree of generation. For those that are finitely generated, find a finite generating set.

Exercise 22. Give an example of each of the following.

(a) An FI–module that does not have finite generation degree

(b) An FI–module that has finite generation degree, but is not finitely generated

Exercise 23. Let $F : V \to W$ be a map of FI–modules, and suppose that V is generated by a set S.

(a) Show that the map F is completely determined by the images $F(S)$.

(b) Show that the FI–module $F(V)$ is generated by $F(S)$.

Exercise 24. (Finiteness properties of the direct sum.)

(a) Suppose that V and W are FI–modules generated in degree $\leq d$. Show that the FI–modules $V \oplus W$ is generated in degree $\leq d$.

(b) Suppose that V and W are finitely generated FI–modules. Show that the FI–modules $V \oplus W$ is finitely generated.

3.2 Representable FI–modules

We know that any R–module is the quotient of a free R–module. We will see that the following special class of FI–modules $M(d)$ play the role of "free" FI–modules.

Definition XI. (Representable FI–modules). Fix a nonnegative integer d. Define the FI–module $M(d)$ by

$$M(d)_n := R \cdot \mathrm{Hom}_{\mathsf{FI}}(d,n) \qquad \text{(the free } R\text{–module on the set } \mathrm{Hom}_{\mathsf{FI}}(d,n))$$

and the action of FI–morphisms by postcomposition. An FI–module of this form is called a *representable* FI–module.

Recall from Exercise 3 that the S_n–representation $M(d)_n$ is isomorphic to the coset representation $R[S_n/S_{n-d}]$.

Exercise 25.

(a) Show that $M(d)$ is generated by the identity morphism $\mathrm{id}_d \in M(d)_d$.

(b) Conclude that if $F : M(d) \to V$ is any map of FI–modules, then F is determined by $F(\mathrm{id}_d)$.

Exercise 26. Show that, as S_n–representations,

$$M(d)_n \cong \mathrm{Ind}_{S_{n-d}}^{S_n} R.$$

Exercise 27. Explicitly describe and compute the decompositions for the rational S_n–representations $M(0)_n$, $M(1)_n$, and $M(2)_n$.

Consider an R–module M generated by a set $S \subseteq M$. Then we know we can realize M as the quotient of the free R–module $R \cdot S$ with basis S,

$$\begin{aligned} R \cdot S &\longrightarrow M \\ 1 \cdot s &\longmapsto s \\ r \cdot s &\longmapsto rs \end{aligned}$$

In the following exercise we will see the analogous property for FI–modules.

Exercise 28. (FI–modules as quotients of representable functors.)

(a) Suppose that an FI–module V is generated by an element $v \in V_d$ for some d. We call such an FI–module *cyclic*. Show that V admits a surjection

$$\begin{aligned} M(d) &\longrightarrow V \\ \mathrm{id}_d &\longmapsto v \\ \alpha &\longmapsto \alpha_*(v) \qquad \text{for any } \alpha \in \mathrm{Hom}_{\mathsf{FI}}([d],[n]). \end{aligned}$$

(b) More generally, suppose that an FI–module V is generated by a set S. Show that V admits a surjection

$$\bigoplus_{d \geq 0} \bigoplus_{\substack{s \in S \\ |s| = d}} M(d) \longrightarrow V$$

where the element id_d in the summand $M(d)$ indexed by $s \in S$ maps to $s \in V_d$.

(c) Show that an FI–module V is generated in degree $\leq d$ if and only if it admits a surjection

$$\bigoplus_{0 \leq m \leq d} M(m)^{\oplus c_m} \longrightarrow V, \qquad c_m \in \mathbb{Z}_{>0} \cup \{\infty\}.$$

(d) Show moreover that V is finitely generated if and only if it admits such a surjection for some $d < \infty$ and with all multiplicities c_m finite.

We will see that we can deduce a lot of combinatorial and representation-theoretic information about finitely generated FI–modules from the structure of the representable functors $M(d)$.

Exercise 29. (Finiteness properties of the pointwise tensor product.)

(a) Show that the tensor product $M(d) \otimes M(k)$ is finitely generated in degree $\leq (d + k)$.

(b) Suppose that V and W are finitely generated FI–modules. Show that the FI–module $V \otimes W$ is finitely generated.

(c) Suppose that V and W are FI–modules generated in degrees $\leq d_V$ and $\leq d_W$, respectively. Show that the FI–modules $V \otimes W$ is generated in degree $\leq (d_V + d_W)$.

3.3 The Noetherian property

The following result is crucial to the study of FI–modules. It is due to Church–Ellenberg–Farb [CEF1, Theorem 1.3] and Snowden [Sn, Theorem 2.3] when R is a Noetherian ring containing $\mathbb{Q}$, and due to Church–Ellenberg–Farb–Nagpal [CEFN, Theorem A] when R is an arbitrary Noetherian ring.

Theorem XII. ***(FI–modules over Noetherian rings are locally Noetherian.)*** *(Church–Ellenberg–Farb [CEF1, Theorem 1.3], Snowden [Sn, Theorem 2.3], Church–Ellenberg–Farb–Nagpal [CEFN, Theorem A].)*
Let R be a commutative Noetherian ring. Then any submodule of a finitely generated FI–module over R is itself finitely generated.

We say that the category of FI–modules over R is *(locally) Noetherian*. Notably, this result means that if $F : V \to W$ is a map of finitely generated FI–modules, then the kernel $\ker(F)$ is finitely generated.

Exercise 30. Let R be a non–Noetherian ring. Show by example that the category of FI–modules over R is not locally Noetherian.

Much of the significance of Theorem XII stems from the following result.

Exercise 31. (Finitely generated FI–modules are finitely presentable and type FP$_\infty$.) Let V be a finitely generated FI–module over a Noetherian ring.

(a) Show that V is *finitely presentable*, in the sense that there exists a short exact sequence of finitely generated FI–modules of the form

$$0 \longrightarrow K \longrightarrow \bigoplus_d M(d)^{c_d} \longrightarrow V \longrightarrow 0$$

(b) Show that V admits a resolution by finitely generated sums of representable functors. In other words, construct an exact sequence of FI–modules of the form

$$\cdots \longrightarrow \bigoplus_d M(d)^{c_d^1} \longrightarrow \bigoplus_d M(d)^{c_d^0} \longrightarrow V \longrightarrow 0$$

where each term $\bigoplus_d M(d)^{c_d^i}$ is finitely generated.

Exercise* 32. Show that the category of FI–modules over $\mathbb{Q}$ is locally Noetherian.
Hint: See Church–Ellenberg–Farb [CEF1, Theorem 1.3].

Exercise* 33. Show that the category of FI–modules over $\mathbb{Z}$ is locally Noetherian.
Hint: See Church–Ellenberg–Farb–Nagpal [CEFN, Theorem A].

4 Induced FI–modules, FI♯–modules, and projective FI–modules

4.1 Induced FI–modules

Recall that the construction of the free R–module on a set S can be viewed as the left adjoint of the forgetful functor from the category of R–modules to the category of sets. Analogously, there are several forgetful functors from the category of FI–modules, whose left adjoint functors can be viewed as "free" constructions, and which play an important role in the theory.

Definition XIII. (The category FB and FB–modules.) Let FB denote the category of finite sets and bijective maps. An FB–*module* over a commutative ring R is a functor from FB to the category of R–modules. A *map of* FB*–modules* is a natural transformation.

Exercise 34. (A skeleton of FB.) Show that the full subcategory of finite sets $[n]$, $n \in \mathbb{Z}_{\geq 0}$, is a skeleton of FB.

Exercise 35. (a) Explain the sense in which an FB–module X is a sequence of S_n–representations X_n, with no additional maps.

(b) Show that a map of FB–modules $F : V \to W$ is a sequence of S_n–equivariant maps $F_n : V_n \to W_n$. What conditions must these maps satisfy?

Exercise 36. (The category of FB–modules.) Fix a commutative ring R. Show that there is a category whose objects are the FB–modules over R and whose morphisms are the FB–module maps.

Definition XIV. (Induced FI–modules.) Fix a commutative ring R. For fixed $d \in \mathbb{Z}_{\geq 0}$, let W_d be a $R[S_d]$–module. Recall from Exercise 4 that for each n the group S_d also acts on $M(d)_n$ on the right. Define an FI–module $M(W_d)$ by

$$M(W_d)_n = M(d)_n \otimes_{R[S_d]} W_d$$

with an action of the FI morphisms on $M(d)_n$ on the left. More generally, if W is an FB–module (that is, a sequence of S_n–representations), define the FI–module $M(W)$ by

$$M(W) = \bigoplus_{d \geq 0} M(W_d).$$

We call FI–modules of this form *induced* FI*–module*, and $M(W)$ the *induced* FI*–module generated by* W.

Notation XV. (External tensor product of representations.) Let $G \times H$ be a product of groups. Recall that, if U is an G–representation over R and W an H–representation over R, we define the $(G \times H)$–representation $U \boxtimes W$ as follows. As an R–module, $U \boxtimes W \cong U \otimes_R W$, and the group $(G \times H)$ acts by

$$\begin{aligned}(g,h) : U \boxtimes W &\longrightarrow U \boxtimes W \\ u \otimes w &\longmapsto (g \cdot u) \otimes (h \cdot w).\end{aligned}$$

Exercise 37. Fix d and let W_d be an $R[S_d]$–module. Show that, as an S_n–representation,

$$M(W_d)_n \cong \mathrm{Ind}_{S_d \times S_{n-d}}^{S_n} W_d \boxtimes R \qquad \text{with } R \text{ the trivial } S_{n-d}\text{–representation.}$$

Exercise 38. Show that the FI morphisms act on $M(W)$ by injective maps $M(W)_m \to M(W)_n$.

Exercise 39. Fix d, and let $R[S_d]$ denote the left regular S_d–representation. Show that there is an isomorphism of FI–modules

$$M(d) \cong M(R[S_d]).$$

Exercise 40. For a FB–module W and a finite set T, show that

$$M(W)_T = \bigoplus_{S \subset T} W_S.$$

Then there is a forgetful functor

$$\mathcal{F} : \mathsf{FI}\text{–Mod} \longrightarrow \mathsf{FB}\text{–Mod}$$

defined by restriction to the subcategory FB $\subseteq$ FI. This forgetful functor takes an FI–module V and remembers only the sequence of $R[S_n]$–modules $\{V_n\}$ and no additional maps. The following exercises show that we may view the assignment $W \mapsto M(W)$ as a functor

$$M(-) : \mathsf{FB}\text{–Mod} \longrightarrow \mathsf{FI}\text{–Mod},$$

and that this functor is a left adjoint to the forgetful functor $\mathcal{F}$.

Exercise 41. ($M(-)$ as a left adjoint.)

(a) Show that the map

$$\begin{aligned} M(-) : \mathsf{FB}\text{–Mod} &\longrightarrow \mathsf{FI}\text{–Mod} \\ W &\longmapsto M(W) \end{aligned}$$

is a covariant functor.

(b) Show that $M(-)$ left adjoint to the forgetful functor $\mathcal{F}$. Concretely, show that for each object $V \in$ FI–Mod and $W \in$ FB–Mod, there is a natural bijection of sets

$$\mathrm{Hom}_{\mathsf{FB}\text{–Mod}}(W, \mathcal{F}(V)) = \mathrm{Hom}_{\mathsf{FI}\text{–Mod}}(M(W), V).$$

(c) Show that the functor $M(-)$ is exact.

Given this adjunction, we may think of $M(W)$ as the FI–module "freely generated" by the sequence of representations $\{W_n\}$.

Exercise 42. (FN–modules.) Let FN be the category whose objects are the sets $[n]$, $n \in \mathbb{Z}_{\geq 0}$, and whose only morphisms are the identity morphisms id_n. An FN–set is a functor from FN to the category of sets, that is, it is a sequence of sets A_n. Then there is a forgetful functor

$$\mathsf{FI}\text{–Mod} \longrightarrow \mathsf{FN}\text{–Set}$$

defined by taking an FI–module V to the underlying sequence of sets. Show that this forgetful functor is the right adjoint to the functor

$$\begin{aligned} \mathsf{FN}\text{–Set} &\longrightarrow \mathsf{FI}\text{–Mod} \\ \{A_n\} &\longmapsto \bigoplus_{d \geq 0} M(d)^{\oplus A_d} \end{aligned}$$

Remark XVI. Some authors refer to FI–modules of the form $\bigoplus_d M(W_d)$ as *free* FI–modules, and some reserve the term *free* for the more restricted class of FI–modules of the form $\bigoplus_d M(d)^{\oplus c_d}$. In these notes we will not enter into this debate, but refer to these FI–modules as *induced* or *sums of representables*, respectively.

4.2 FI♯–modules

Definition XVII. (Based sets and maps of based sets.) A *based set* S_0 is a set with a distinguished element $0 \in S_0$, called the *basepoint*. A map of based sets $F : S_0 \to T_0$ is a map of sets that takes the basepoint in S_0 to the basepoint in T_0.

Definition XVIII. (The category FI♯) Let FI♯ (read "FI–sharp") be the category defined as follows. The objects are finite based sets. The morphisms are maps of based sets that are injective away from the basepoints, in the following sense. If $f : S_0 \to T_0$ is map of based sets, then f is an FI♯ morphism if $f^{-1}(t)$ has cardinality $|f^{-1}(t)| \leq 1$ for all $t \in T_0$ not equal to the basepoint.

Notation XIX. For $n \in \mathbb{Z}_{\geq 0}$, let $[n]_0$ denote the based set

$$[n]_0 := \{0, 1, 2, \dots, n\} \qquad \text{with basepoint } 0.$$

Exercise 43.

(a) Show that FI♯ is isomorphic to its opposite category FI♯^{op}.

(b) Show that $S_n \subsetneq \mathrm{End}_{\mathsf{FI}\sharp}([n]_0)$, but that S_n is exactly the group of invertible endomorphisms of the object $[n]_0$.

(c) Describe an embedding FI $\subseteq$ FI♯.

(d) Show that the image of every FI morphism in FI♯ has a one-sided inverse.

Exercise 44. (An alternate description of FI♯.) Show that FI♯ is isomorphic to the following category, which was the original description given by Church–Ellenberg–Farb [CEF1, Definition 4.1.1]. The objects are finite sets. The morphisms $\mathrm{Hom}(S, T)$ are triples (A, B, α) with $A \subseteq S$, $B \subseteq T$, and $\alpha : A \to B$ a bijection. The composition of morphisms $(A, B, \alpha) : S \to T$ and $(D, E, \delta) : T \to U$ is the morphism

$$(\alpha^{-1}(B \cap D), \delta(B \cap D), \delta \circ \alpha) : S \to U.$$

Definition XX. (FI♯–modules.) An FI♯–module over a commutative ring R is a functor from FI♯ to the category of R–modules.

An FI^{op}*–module* over a ring R is a functor from the opposite category FI^{op} of FI to the category of R–modules. Equivalently, it is a contravariant functor from FI to R–modules. In the following exercise we will see that an FI♯–module simultaneously carries an FI– and an FI^{op}–module structure in a compatible way.

Exercise 45. Show that any FI♯–module is both an FI–module and an FI^{op}–module. Describe what relations must be satisfied by the actions of the FI morphisms and FI^{op} morphisms.

Exercise 46. Let W_d be an $R[S_d]$–module. Show that the FI–module structure on $M(W_d)$ can be promoted to an FI♯–module structure.

The following exercise gives a complete characterization of FI♯–modules. It is a result of Church–Ellenberg–Farb [CEF1, Theorem 4.1.5], and it mirrors an earlier result of Pirashvili [Pi, Theorem 3.1].

Exercise* 47. (The structure of the category of FI♯–modules.)
Hint: See Church–Ellenberg–Farb [CEF1, Theorem 4.1.5].

(a) Show that every FI♯–module has the form $M(W)$ for some FB–module W.

(b) Show that the functor
$$M(-) : \text{FB–Mod} \longrightarrow \text{FI}\sharp\text{–Mod}$$
is an equivalence of categories.

Exercise 48. Let R be a field of characteristic zero. Conclude from Exercise 47 that the category of FI♯–modules over R is semisimple.

Exercise 49. (Polynomial and exterior algebras as FI♯–modules.)

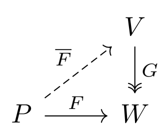
(a) Let V be the FI–module with $V_n = \mathbb{Z}[x_1, \ldots, x_n]$ and inclusions $V_n \hookrightarrow V_{n+1}$. Show that V is an FI♯–module.

(b) Consider the FI♯–submodules of V consisting of homogeneous degree k polynomials for $k = 0, 1, 2, 3$. Explicity write each of these FI♯–modules in the form $\bigoplus_{d \geq 0} M(W_d)$ for appropriate S_d–representations W_d.

(c) Repeat these exercises for the case that V is the sequence of exterior algebras $V_n = \bigwedge_{\mathbb{Z}}\langle x_1, \ldots, x_n \rangle$.

4.3 Projective FI–modules

Definition XXI. (Projective FI–modules.) An FI–module P is *projective* if it is a projective object in the abelian category of FI–modules. Recall that this means that for any surjective map of FI–modules $G : V \to W$ and map of FI–modules $F : P \to W$, there is a lift $\overline{F} : P \to V$ making the following diagram commute.



Exercise 50. (Projective FI–modules.)

(a) Suppose that W_d is a projective $R[S_d]$–module. Show that $M(W_d)$ is a projective FI–module. *Hint:* By Weibel [We, Proposition 2.3.10], it suffices to show that $M(-)$ is the left adjoint to an exact functor. See Exercise 41.

(b) Show that the projective FI–modules are precisely the FI–modules of the form $\bigoplus_d M(W_d)$ for projective $R[S_d]$–modules W_d.

(c) Conclude that if R is a field of characteristic zero, an FI–module is projective if and only if it is an FI♯–module.

4.4 $\sharp$–filtered FI–modules

Nagpal [Na1] intrroduced the following definition.

Definition XXII. ($\sharp$–filtered FI–modules.) (Nagpal [Na1, Definition 1.10].) A $\sharp$–filtered FI–module V is a surjection

$$\Pi : \bigoplus_{i=1}^{d} M(d_i) \longrightarrow V$$

of FI–modules such that the filtration

$$V^1 \subseteq V^2 \subseteq \cdots \subseteq V^d = V \qquad\qquad V^r := \Pi\left(\bigoplus_{i=1}^{r} M(d_i)\right)$$

has graded pieces of the form $V_r/V_{r-1} \cong M(W_r)$ for some $R[S_{d_r}]$–modules W_r.

> **Exercise 51.** Show that if $R = \mathbb{Q}$, then $\sharp$–filtered FI–modules are precisely the FI$\sharp$–modules.
>
> **Exercise 52.** Find an example of an FI–module that is not $\sharp$–filtered.

5 Multiplicity stability and character polynomials

The following Theorem XXIII and Theorem XXIV summarize the stability results for finitely generated FI–modules proved by Church–Ellenberg–Farb [CEF1, CEF2]. In this section, we will define the terms in these theorems and investigate the combinatorics of the irreducible constituents and characters of finitely generated FI–modules.

Theorem XXIII (Church–Ellenberg–Farb [CEF1, CEF2]). *Let V be an FI–module over $\mathbb{Q}$ that is finitely generated in degree $\leq d$. Then the following hold.*

- ***(Multiplicity stability).*** *The decomposition of V_n into irreducible representations is independent of n for all n sufficiently large.*
- ***(Polynomial dimension growth).*** *The dimensions $\dim_{\mathbb{Q}}(V_n)$ are, for n sufficiently large, equal to the integer points $p(n)$ of a polynomial p of degree $\leq d$.*
- ***(Polynomial characters).*** *For all n sufficiently large, the sequence of characters χ_{V_n} are equal to a character polynomial P that is independent of n;*

$$\chi_n(\sigma) = P(\sigma) \qquad\qquad \textit{for all } \sigma \in S_n \textit{ and all } n \textit{ sufficiently large.}$$

- ***(Stable inner products).*** *If Q is any character polynomial, then $\langle \chi_{V_n}, Q\rangle_{S_n}$ is independent of n for all n sufficiently large.*
- ***(Finite presentability).*** *V is* finitely presentable *as an FI–module.*

In the case that V is has an FI$\sharp$–module structure, Church–Ellenberg–Farb obtained the following strengthened results.

Theorem XXIV (Church–Ellenberg–Farb [CEF1, CEF2]). *Let V be an FI$\sharp$–module over $\mathbb{Z}$ that is finitely generated as an FI–module in degree $\leq d$. Then the following hold.*

- ***(Multiplicity stability).*** *The decomposition of $\mathbb{Q} \otimes_{\mathbb{Z}} V_n$ into irreducible representations is independent of n for all $n \geq 2d$.*
- ***(Polynomial dimension growth).*** *For all n the ranks $\text{rank}_{\mathbb{Z}}(V_n)$ are equal to the integer points $p(n)$ of a polynomial p of degree $\leq d$ that is independent of n.*
- ***(Polynomial characters).*** *The sequence of characters $\chi_{\mathbb{Q}\otimes_{\mathbb{Z}} V_n}$ are equal to a character polynomial P that is independent of n.*
- ***(Stable inner products).*** *If Q is any character polynomial, then $\langle \chi_{\mathbb{Q}\otimes_{\mathbb{Z}} V_n}, Q\rangle_{S_n}$ is independent of n for all $n \geq (d + \deg(Q))$.*
- ***(Structure theorem).*** *For $m = 0, \ldots, d$ there are S_m–representations U_m such that*

$$V_n \cong \bigoplus_{m=0}^{d} \mathit{Ind}_{S_m \times S_{n-m}}^{S_n} U_m \boxtimes \mathbb{Z} \qquad \mathbb{Z} \text{ the trivial } S_{n-m}\text{–representation}$$

  and morphisms act by the natural injective maps $V_n \to V_{n'}$.

The structure theorem is a restatement of Exercise 47.

A main ingredient in the proof of these results is an analysis of the representations occurring in the representable functors $M(d)$. The proof also uses the Noetherian property of FI–modules, and in particular the finite presentability result Exercise 31(a).

5.1 Multiplicity stability

Recall that every rational representation of the symmetric groups S_n can be decomposed as a direct sum of irreducible representations, and the irreducible rational S_n–representations are in canonical bijection with the set of partitions λ of n.

Notation XXV. Given a partition λ of n, we write V_λ for the associated irreducible S_n–representation.

Church–Ellenberg–Farb proved that for any finitely generated FI–module V over $\mathbb{Q}$, the decomposition of the S_n–representations V_n into their irreducible constituents in a sense stabilizes as n tends to infinity. To make sense of this result, we use the following 'stable' notation for irreducible S_n–representations, which implicitly assembles irreducible S_n–representations into families as n varies.

Notation XXVI. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ be a partition with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. Let

$$|\lambda| := \lambda_1 + \cdots + \lambda_k.$$

Then for $n \geq |\lambda| + \lambda_1$, we write $\lambda[n]$ for the partition of n

$$\lambda[n] := (n - |\lambda|, \lambda_1, \lambda_2, \cdots, \lambda_k).$$

Note that every partition of n can be written uniquely in this form. We write $V(\lambda)_n$ for the S_n–representation

$$V(\lambda)_n := \begin{cases} 0, & n < |\lambda| + \lambda_1 \\ V_{\lambda[n]}, & n \geq |\lambda| + \lambda_1 \end{cases}$$

Definition XXVII. (Multiplicity stability.)(Church–Farb [CF, Definition 1.1(III)].) Let V_n be a sequence of rational S_n–representations, with decomposition into irreducible constituents

$$V_n = \bigoplus_{\lambda} c_{\lambda}^{n} V(\lambda)_n.$$

Then V_n is called *(uniformly) multiplicity stable* if there exists some $N \geq 0$ such that, for all λ and for all $n \geq N$, the multiplicities $c_{\lambda}^{n} = c_{\lambda}^{N}$ are independent of n.

A sequence V_n of S_n–representations is multiplicity stable if, for n sufficiently large, we can determine the decomposition of V_{n+1} from that of V_n by simply adding a single box to the top row of the Young diagrams corresponding to each irreducible constituent of V_n.

Exercise 53. Stable and unstable sequences of S_n–representations

(a) Confirm that the following sequences of S_n–representations are multiplicity stable.
   (i) $V_n = \mathbb{Q}$ the trivial S_n–representations. Show that $V_n = V(0)_n$ for all $n \geq 0$.
   (ii) $V_n = \mathbb{Q}^n$ the canonical permutation representation. Show that $V_n = V(0)_n \oplus V(1)_n$ for all $n \geq 0$.
   (iii) $V_n = \bigwedge^k V(1)_n$, with $V(1)_n$ the standard representation. Show that $V_n = V(1^k)_n$ for all $n \geq 0$.
   (iv) $V_n = \mathbb{Q}[x_1, \ldots, x_n]_{(k)}$ with S_n permuting the indices. Decompose V_n for $k = 2, 3$.

(b) Show that the following sequences are *not* multiplicity stable.
   (i) $V_n = \mathbb{Q}$ the alternating S_n–representations.
   (ii) $V_n = \mathbb{Q}[S_n]$ the left regular representation.

Definition XXVIII. (Weight.) The *weight* of an irreducible representation $V(\lambda)_n$ is the quantity $|\lambda|$. The *weight* of a general S_n–representation V_n is the maximum weight of its irreducible constituents.

Church–Ellenberg–Farb proved the following.

Theorem XXIX. ***(Multiplicity stability and weight of finitely generated* FI*–modules.)***
(Church–Ellenberg–Farb [CEF1, Theorem 1.13 and Proposition 3.2.5].)
An FI*–module V over a field of characteristic 0 is finitely generated if and only if the sequence $\{V_n\}$ of S_n–representations is uniformly representation stable and each V_n is finite-dimensional. If V is generated in degree $\leq d$, then V_n has weight $\leq d$ for all n.*

We can use the *branching rules* for the symmetric group to verify these stability patterns for induced FI–modules. Below we state a special case, Pieri's rule.

Theorem XXX. ***(Pieri's rule.)***

(I) Let μ be a partition of k. Let $\mathbb{Q}$ denote the trivial S_{n-k}–representation. Then, as rational S_n–representations, there is a decomposition

$$Ind_{S_k \times S_{n-k}}^{S_n} V_{\mu} \boxtimes \mathbb{Q} = \bigoplus_{\lambda} V_{\lambda}$$

where the sum is taken over all those partitions λ that can be obtained by adding $(n-k)$ boxes to the Young diagram for μ, each in a distinct column. Each component V_{λ} occurs with multiplicity 1.

(II) *Let λ be a partition of n. Then the S_{n-k}–coinvariants $\left(Res^{S_n}_{S_k \times S_{n-k}} V_\lambda\right)_{S_{n-k}}$ of the restricted representation $\left(Res^{S_n}_{S_k \times S_{n-k}} V_\lambda\right)$ are a representation of S_k, with the following decomposition*

$$\left(Res^{S_n}_{S_k \times S_{n-k}} V_\lambda\right)_{S_{n-k}} = \bigoplus_{\mu} V_\mu.$$

The sum is over those partitions μ that can be obtained from λ by removing $(n-k)$ boxes, each from a distinct column. The component V_μ occurs with multiplicity 1.

Example XXXI. (Decomposing $M\left(V_{(2,1)}\right)$.) Consider, for example, the FI–module $M\left(V_{(2,1)}\right)$. By Exercise 37,

$$M\left(V_{(2,1)}\right)_n \cong \mathrm{Ind}^{S_n}_{S_3 \times S_{n-3}} V_{(2,1)} \boxtimes \mathbb{Q}$$

and this decomposition can be described by Pieri's rule in Theorem XXX. We find

$$\begin{aligned}
M\left(V_{(2,1)}\right)_0 &= 0 \\
M\left(V_{(2,1)}\right)_1 &= 0 \\
M\left(V_{(2,1)}\right)_2 &= 0 \\
M\left(V_{(2,1)}\right)_3 &= V_{(2,1)} \\
M\left(V_{(2,1)}\right)_4 &= V_{(3,1)} \oplus V_{(2,2)} \oplus V_{(2,1,1)} \\
M\left(V_{(2,1)}\right)_5 &= V_{(4,1)} \oplus V_{(3,2)} \oplus V_{(3,1,1)} \oplus V_{(2,2,1)} \\
M\left(V_{(2,1)}\right)_6 &= V_{(5,1)} \oplus V_{(4,2)} \oplus V_{(4,1,1)} \oplus V_{(3,2,1)} \\
M\left(V_{(2,1)}\right)_7 &= V_{(6,1)} \oplus V_{(5,2)} \oplus V_{(5,1,1)} \oplus V_{(4,2,1)} \\
&\vdots
\end{aligned}$$

Observe that the sequence $M\left(V_{(2,1)}\right)_n$ is (uniformly) representation stable, stabilizing for $n \geq 5$.

Exercise 54. (The decomposition of $M(W)$.) . Let W_d be a rational S_d–representation. Recall from Exercise 37 that

$$M(W_d)_n \cong \mathrm{Ind}^{S_n}_{S_d \times S_{n-d}} W_d \boxtimes \mathbb{Q} \qquad \text{with } \mathbb{Q} \text{ the trivial } S_{n-d}\text{–representation.}$$

Use Pieri's rule (Theorem XXX) to do the following.

(a) Show that $M(W_d)_n$ has weight $\leq d$ for all n.

(b) Show that an irreducible representation occurs in $M(d)_n$ if and only if it has weight $\leq d$.

(c) Show that $\{M(W_d)_n\}$ is (uniformly) multiplicity stable, stabilizing for all $n \geq 2d$.

(d) Give a sharp stable range for $\{M(V_\lambda)_n\}$ in terms of the shape of λ.

(e) Suppose V is a rational FI–module generated in degree $\leq d$. Show that V_n has weight $\leq d$ for all n.

Nagpal proved the following result [Na1, Theorem A].

Theorem XXXII. ***(A resolution in a range.)*** *Let V be a finitely generated* FI*–module over a Noetherian ring R. Then there is a sequence of* FI*–modules*

$$0 \longrightarrow V \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots \longrightarrow J^N \longrightarrow 0$$

that is exact for all n sufficiently large, and such that the FI*–modules J^i are $\sharp$–filtered in the sense of Definition XXII.*

Exercise 55. Suppose that V is a finitely generated FI–module over a field of characteristic zero. Use Nagpal's Theorem XXXII to show that $\{V_n\}$ is multiplicity stable.

5.2 An application to classical representation theory: Murnaghan's theorem

Church–Ellenberg–Farb [CEF1, Theorem 3.4.2] observed that these stability results for FI–modules give an efficient proof of Murnaghan's classical result on Kronecker coefficients. Given any partitions μ and λ, the structure constants $g^{\nu}_{\mu,\lambda}(n)$ in the tensor product

$$V(\mu)_n \otimes_{\mathbb{Q}} V(\lambda)_n = \bigoplus_{\nu} g^{\nu}_{\mu,\lambda}(n) V(\nu)_n$$

are called *Kronecker coefficients.* Values of these coefficients are unknown in general, but Littlewood [L] proved that the multiplicity $g^{\nu}_{\mu,\lambda}(n)$ is independent of n for all n sufficiently large. In the context of FI–modules we can interpret this result as the statement of finite generation for the tensor product of finitely generated FI–modules.

Exercise 56. Murnaghan's Theorem for Kronecker coefficients

(a) Fix a partition λ. Show that the sequence of S_n–representations $V(\lambda)_n$ has the structure of a finitely generated FI–module.
Hint: Use the branching rules to realize $V(\lambda)$ as an FI–submodule of $M(V_\lambda)$. See [CEF1, Proposition 3.4.1].

(b) Given partitions μ and λ, use Exercise 29 to conclude that $V(\mu) \otimes_{\mathbb{Q}} V(\lambda)$ is a finitely generated FI–module, and therefore multiplicity stable.

5.3 Character polynomials

One of the main results of Church–Ellenberg–Farb [CEF1] is the statement that the characters of a finitely generated FI–module agree, for n sufficiently large, with an object called a character polynomial.

Definition XXXIII. (Character polynomials.) Let $X_r : \coprod_{n\geq 0} S_n \to \mathbb{Z}$ denote the class function

$$X_r(\sigma) = \#r\text{–cycles in the cycle type of } \sigma.$$

A polynomial $P \in \mathbb{Q}[X_1, X_2, \ldots, X_r, \ldots]$ is called a *character polynomial*. Such a polynomial P defines a sequence of class functions on S_n for all $n \geq 0$. We define the *degree* of a character polynomial by setting $\deg(X_r) = r$.

Character polynomials and their role in algebraic combinatorics have been studied since Murnaghan [Mu] and Specht [Sp]. See Macdonald [Ma, Section I.7].

Example XXXIV. (The characters of $\mathbb{Q}[x_1, x_2, \ldots x_n]_{(2)}$.) Consider again the FI–module V of homogeneous degree–2 polynomials in the variables $x_1, \ldots, x_n$ over $\mathbb{Q}$, with inclusions. The S_n–representation V_n has a $\mathbb{Q}$–basis

$$\{ \; x_i^2 \mid i \in [n] \; \} \sqcup \{ \; x_i x_j \mid i < j, \; i, j \in [n] \; \}.$$

Since S_n acts by permuting this basis, the character χ_{V_n} is equal to the number of basis elements fixed by this action. Observe that a permutation σ will fix x_i^2 exactly when its cycle decomposition contains the 1–cycle (i), and σ will fix $x_i x_j$ if it contains $(i \; j)$ or $(i)(j)$. Hence,

$$\chi_{V_n}(\sigma) = (\#1\text{–cycles of } \sigma) + (\#2\text{–cycles of } \sigma) + \binom{\#1\text{–cycles of } \sigma}{2}$$

Thus the sequence of characters $\{\chi_{V_n}\}$ is exactly described by the character polynomial

$$\chi_{V_n} \;\; = \;\; X_1 + X_2 + \binom{X_1}{2} \;\; = \;\; X_1 + X_2 + \frac{X_1(X_1 - 1)}{2} \qquad \text{for all } n \geq 0.$$

Observe that this character polynomial is independent of n.

Exercise 57. Examples of character polynomials

(a) Compute the character polynomials associated with the following FI–modules, and determine their degrees. All FI–morphisms act by the natural injective maps.
   - (i) V_n is the space $\mathbb{Q}[x_1, \ldots, x_n]_{(k)}$ of homogeneous degree-k polynomials, for $k = 0, 1, 2, 3, 4$.
   - (ii) V_n is the k^{th} exterior power $\bigwedge_{\mathbb{Q}}^k (x_1, \ldots, x_n)$, for $k = 0, 1, 2, 3, 4$.
   - (iii) V_n is the standard representations $V(\square)_n$.
   - (iv) V is the induced FI–module $M(V_{\square})$.

(b) Show that the FI–module $M(d)$ has characters given by the degree-d character polynomial

$$\chi_{M(d)} = d! \binom{X_1}{d} \qquad \text{for all } n \geq 0$$

Theorem XXXV. ***(Finitely generated* FI*–modules have eventually polynomial characters.)*** *(Church–Ellenberg–Farb [CEF1, Theorem 3.3.4].) Let V be an* FI*–module over a field of characteristic 0 finitely generated in degree $\leq d$. Then there an integer N and a unique character polynomial $P(X_1, \ldots, X_d)$ of degree $\leq d$ such that*

$$\chi_{V_n}(\sigma) = P(X_1, \ldots, X_d)(\sigma) \qquad \textit{for all } n \geq N \textit{ and all } \sigma \in S_n.$$

The following exercise will establish that an FI♯–module generated in degree $\leq d$ has a character polynomial of degree $\leq d$.

Exercise 58. (Character polynomials of rational FI♯–modules.) Given a partition ρ, let $n_r(\rho)$ denote the number of parts of ρ of length r. We use *generalized binomial coefficients* to define the character polynomials

$$\binom{\mathbf{X}}{\rho} := \prod_r \binom{X_r}{n_r(\rho)} = \prod_r \frac{X_r(X_r - 1) \cdots (X_r - n_r(\rho) + 1)}{n_r(\rho)!}$$

For a representation U of S_d, let χ^U denote its character. For a partition α of d let χ^U_α denote the value of χ^U on a permutation of cycle type α.

(a) Show that the induced representation $\mathrm{Ind}_{S_d \times S_{n-d}}^{S_n} U \boxtimes U'$ has character

$$\chi(\sigma) = \sum_{\text{partitions } \alpha \text{ of } n} \chi_\alpha^U \chi_\beta^{U'} \binom{\mathbf{X}}{\alpha}(\sigma).$$

Here, β is chosen so that the partition $\beta \sqcup \alpha$ is the cycle type of σ. If no such decomposition of its cycle type exists, the term $\binom{\mathbf{X}}{\alpha}(\sigma)$ will be zero.

(b) Let W_d be an S_d–representation. Deduce that the characters of $M(W_d)$ are equal to a character polynomial of degree $\leq d$.

Exercise 59. (Polynomial growth of dimension.)

(a) Let V_n be an S_n–representation over $\mathbb{Q}$. Show that

$$\dim_{\mathbb{Q}}(V_n) = \chi_n(\mathrm{id}_n).$$

(b) Show that $X_r(\mathrm{id}_n) = \begin{cases} n, & r = 1 \\ 0, & r \neq 1. \end{cases}$

(c) Let V be an FI–module over $\mathbb{Q}$ finitely generated in degree $\leq d$. By Theorem XXXV, for n sufficiently large, the characters of V_n coincide with a character polynomial P_V of degree $\leq d$. Prove that the dimensions $\dim_{\mathbb{Q}}(V_n)$ are, for n large, equal to the integer values of a polynomial $p(n)$ of degree $\leq d$.

Nagpal's Theorem XXXII implies the polynomiality result for FI–modules over general Noetherian commutative rings.

Exercise 60. (Polynomial growth of dimension.) Suppose that V is a finitely generated FI–module over a Noetherian ring R. Use Theorem XXXII to show that the ranks $\mathrm{rank}_R(V_n)$ are, for n large, equal to the integer values of a polynomial $p(n)$.

6 Presentation degree, polynomial degree, and central stability degree

In this section, we survey some different notions of stability for FI–modules, including the finite presentation degree, finite polynomial degree, and central stability. These concepts turn out to be equivalent for FI–modules, although this is not always in the case for other functor categories that generalize FI–modules.

6.1 Projective resolutions and finite presentation degree

Definition XXXVI. (Relation degree and presentation degree.) Let V be an FI–module. Suppose that V admits a partial resolution by projective FI–modules

$$P^1 \longrightarrow P^0 \longrightarrow V \longrightarrow 0.$$

We have previously seen that if P^0 is generated in degree $\leq d$, then d is a bound on the generation degree of V. If P^1 is generated in degree $\leq r$ when we say that V is *related in degree* $\leq r$. We say that V is *presented in degree* $\leq \max(r, d)$. If P^1 and P^0 are both finitely generated FI–modules, then we say that V is *finitely presented*.

Note that being 'finitely presented' is a stronger condition than being presented in finite degree. Recall that the Noetherian property Theorem XII for FI–modules implies that an FI–module is finitely generated if and only if it is finitely presented, though this equivalence does not hold for other important classes of functor categories.

Exercise 61. Projective dimension

(a) Let V be an FI–module over $\mathbb{Q}$, and suppose that V admits a finite projective resolution. Explain why the characters of V must be equal a character polynomial identically for all $n \geq 0$.

(b) Show by example that there are finitely generated FI–modules that do not admit finite projective resolutions.

Exercise 62. An example of a projective resolution Let V be the torsion FI–module with

$$V_n = \begin{cases} \mathbb{Q}, & n = 0 \\ 0, & n \geq 0 \end{cases}$$

Compute a resolution of V_n by induced FI–modules.

Exercise* 63. (FI–modules with finite presentation degree in their stable range.) Suppose that V is an FI–module presented in degree $\leq d$. Show that, for all $n > d$,

$$V_n = \operatorname{colim}_{\substack{S \subseteq [n] \\ |S| \leq d}} V_S.$$

Hint: See subsection 6.3 and Church–Ellenberg–Farb–Nagpal [CEFN, Theorem C].

6.2 Shifts, derivatives, and polynomial functors

Our next notion of stability for an FI–module is a polynomiality condition. The version of this definition used here was called a *degree-r coefficient system* by Randal-Williams–Wahl [RWW]; this is a closely related and slightly stronger condition than that of *strong polynomial degree* of Djament and Vespa [DV, Dj2].

Definition XXXVII. (The shift functor.) The category FI has a proper self-embedding defined by

$$\begin{aligned} \overset{\star}{\amalg} : \mathsf{FI} &\longrightarrow \mathsf{FI} \\ S &\longmapsto S \sqcup \{\star\} \end{aligned}$$

A morphism $f : S \to T$ extends to a morphism $f_\star : S \sqcup \{\star\} \to T \sqcup \{\star\}$ with $f_\star(\star) = \star$. We can then define the *shift functor* $\overset{\star}{\Sigma}$ on FI–modules by

$$\overset{\star}{\Sigma} : \mathsf{FI}\text{–Mod} \longrightarrow \mathsf{FI}\text{–Mod}$$

$$[V : \mathsf{FI} \to R\text{–Mod}] \longmapsto [V \circ \overset{\star}{\amalg} : \mathsf{FI} \to R\text{–Mod}]$$

Concretely,

$$(\overset{\star}{\Sigma} V)_n \cong \operatorname{Res}^{S_{n+1}}_{S_n} V_{n+1}.$$

We write $\overset{\star}{\Sigma}{}^b$ for the b^{th} iterate of $\overset{\star}{\Sigma}$.

We analogously define a functor $\overset{\star}{\Sigma}$ on the category of FB–modules.

Exercise 64. (Shifts and finite generation.) Let V be an FI–module.

(a) Show that, if V is generated in degree $\leq d$, then $\overset{\star}{\Sigma}^b V$ is generated in degree $\leq d$.

(b) Show that, if $\overset{\star}{\Sigma}^b V$ is generated in degree $\leq d$, then V is generated in degree $\leq (b+d)$.

Definition XXXVIII. (The derivative functor.) There is a natural transformation

$$\begin{aligned} \mathrm{id}_{\mathsf{FI}} &\longrightarrow \overset{\star}{\amalg} \\ \iota_S : S &\hookrightarrow S \sqcup \{\star\} \end{aligned}$$

which induces a natural transformation

$$\begin{aligned} \mathrm{id}_{\mathsf{FI\text{–}Mod}} &\longrightarrow \overset{\star}{\Sigma} \\ V &\longrightarrow \overset{\star}{\Sigma} V \\ (\iota_S)_* : V_S &\longrightarrow V_{S \sqcup \{\star\}} \end{aligned}$$

The *derivative functor* D is a functor from FI–Mod to FI–Mod that takes an FI–module to its cokernel

$$DV := \mathrm{coker}(V \longrightarrow \overset{\star}{\Sigma} V).$$

Exercise 65. (The derivative of a representable functor.) Compute the FI–modul $\overset{\star}{\Sigma} M(d)$. Show in particular that

$$\overset{\star}{\Sigma} M(d) \cong M(d) \oplus W,$$

where W is a sum of representable functors generated in degree $< d$. What can you deduce about $DM(d)$?

Exercise 66. (The derivative of an induced module.) Show that there is a natural isomorphism of functors

$$D \circ M(-) = M(\overset{\star}{\Sigma} -) : \mathsf{FB\text{–}Mod} \to \mathsf{FI\text{–}Mod}.$$

Definition XXXIX. (Polynomial degree.) Let V be an FI–module. We say that V has polynomial degree -1 in FI–degree $> N$ if $V_n = 0$ for all $n > N$. For $k \geq 0$, we say that V has polynomial degree $\leq k$ in FI–degree $> N$ if

- $V_n \to \overset{\star}{\Sigma} V_n$ is injective for all $n > N$, and
- DV has polynomial degree $\leq (k-1)$ in FI–degree $> N$.

We say V has polynomial degree $\leq k$ if it has polynomial degree $\leq k$ in all FI–degree > -1.

Exercise 67. Show that the analogous inductive statement does indeed characterize polynomials. Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a function. We say that f has degree -1 if $f(n) = 0$ for all n, and we say that f is polynomial of degree $\leq k$ if the function $(f(n) - f(n-1))$ is polynomial of degree $\leq (k-1)$.

Exercise 68. (Polynomial degree of induced modules.)

(a) Show that $M(d)$ has polynomial degree d.

(b) Let W_d be an S_d–representation. Show that $M(W_d)$ has poylnomial degree $\leq d$. Must it have polynomial degree exactly d?

For an FI–module, finite polynomial degree is equivalent to finite presentation degree. This is not always the case, however, for the different functor categories that generalize FI–modules.

Exercise* 69. *Hint:* See Djament [Dj2, Proposition 4.4].

(a) **(Finite polynomial degree is equivalent to finite generation degree.)**

   (i) Suppose that V is an FI–module finitely generated in degree $\leq d$. Show that, in sufficiently large FI–degree, V has polynomial degree $\leq d$.

   (ii) Suppose that V is an FI–module of polynomial degree $\leq r$ in FI–degree $> N$. Then V is generated in degree $\leq r + N + 1$.

(b) **(Polynomial degree is equivalent to finite presentation degree.)**

   (i) Suppose that V is an FI–module generated in degree $\leq d$ and presented in degree $\leq k$. Then V has polynomial degree $\leq d$ in FI–degree $> k + \min(k, d) - 1$.

   (ii) Suppose that V is an FI–module of polynomial degree $\leq r$ in FI–degree $> N$. Then V is presented in degree $\leq r + N + 2$.

6.3 Central stability

A fruitful approach to proving representation stability results in geometry and topology has been through an analysis of certain *functor homology groups* associated to FI–modules. In this section we will describe one such construction, the homology of chain complexes that are sometimes called the *central stability chain complexes* in the literature. Note that its homology differs slightly from the FI homology groups that are used, for example, by [Pu, CEFN, CE]. The name "central stability chain complex" is a tribute to Putman [Pu], though the chain groups he calls by this name are in fact quotients of the one defined below. *Central stability degree* in the sense stated below is equivalent to the corresponding definition in terms of Putman's chain complex (see for example [Pa, Proposition 6.2]).

Definition XL. (The central stability chain complex.) Given an FI–module V, define an associated augmented chain complex of FI–modules by

$$\widetilde{C}_{-1}(V)_n = V_n$$

$$\widetilde{C}_p(V)_n = \bigoplus_{\text{injections } f:[p+1]\hookrightarrow[n]} V_{[n]\setminus \mathrm{im}(f)}$$

$$\cong \mathrm{Ind}_{S_{n-(p+1)}}^{S_n} V_{n-(p+1)}$$

with differential

$$d : \widetilde{C}_p(V) \to \widetilde{C}_{p-1}(V)_n$$

$$d = \sum_{i=1}^{p+1} (-1)^{i+1} d_i$$

where

$$d_i : \bigoplus_{f:[p+1]\hookrightarrow[n]} V_{[n]\setminus \mathrm{im}(f)} \longrightarrow \bigoplus_{\overline{f}=f|_{[p+1]\setminus\{i\}}} V_{[n]\setminus \mathrm{im}(\overline{f})}$$

is defined by forgetting the element i from the domain of the injective map f, and using the maps

$$V_{[n]\setminus \mathrm{im}(f)} \longrightarrow V_{[n]\setminus \mathrm{im}(\overline{f})}$$

induced by the inclusion of sets

$$\Big([n]\setminus \mathrm{im}(f)\Big) \hookrightarrow \Big([n]\setminus \mathrm{im}(\overline{f})\Big) = \Big([n]\setminus \mathrm{im}(f)\Big) \cup \{f(i)\}.$$

This chain complex is significant in that its homology groups in degrees -1 and 0 govern the generation and relation degree of the FI–module V. We will see that

$$\widetilde{H}_{-1}(V)_n = \widetilde{H}_0(V)_n = 0 \quad \text{for } n >> 0 \quad \Longleftrightarrow \quad V \text{ has small presentation degree.}$$

Remark XLI. The homology groups $\widetilde{H}_{-1}(V)$ is denoted by $H_0(V)$ by Church–Ellenberg–Farb [CEF1], and viewed as a functor FI–Mod $\to$ FB–Mod. The indexing convention used here is natural if we view our chain complex as arising from a semi-simplicial object. Church–Ellenberg [CE] and others study the derived functors of the functor $H_0(-)$, which are closely related but nonisomorphic in general to the homology groups $\widetilde{H}_p(-)$ defined above.

Exercise 70. (Properties of $\widetilde{H}_{-1}(-)$.)

(a) Show that the tail of the central stability chain complex is as follows.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \widetilde{C}_0(V)_n & \longrightarrow & \widetilde{C}_{-1}(V)_n & \longrightarrow & 0. \\ & & || & & || & & \\ & & \mathrm{Ind}_{S_{n-1}}^{S_n} V_{n-1} & & V_n & & \end{array}$$

(b) Show that

$$\widetilde{H}_{-1}(V)_n = 0 \quad \text{for } n > d$$

if and only if V is generated in degree $\leq d$.

(c) Viewing $\widetilde{H}_{-1}(-)$ as a functor FI–Mod to FB–Mod, show that

$$\widetilde{H}_{-1}(M(W)) = W,$$

and that $\widetilde{H}_{-1}(-)$ and $M(-)$ are inverse functors FI♯–Mod $\cong$ FB–Mod.

(d) Explain the sense in which the groups $\widetilde{H}_{-1}(V)_n$ are encoding (a quotient of) a minimal set of $R[S_n]$–modules that generate V.

(e) Viewing $\widetilde{H}_{-1}(-)$ as a functor FI–Mod to FB–Mod, show that $\widetilde{H}_{-1}(-)$ is right-exact. We can therefore define its left derived functors.

Exercise 71. (The central stability homology of $M(0)$.) In the case $V = M(0)$, the central stability complex is a chain complex of independent interest in algebraic topology. It is the chain complex on the *complex of injective words*.

(a) Give an explicit combinatorial description of the chain complex $\widetilde{C}_*(M(0))$.

(b)* Show that the homology groups $\widetilde{H}_p(M(0))_n$ are nonzero only when $p = n-1$. In other words, for n fixed the reduced chain complex $\widetilde{C}_*(M(0))_n$ has the homology of a wedge of spheres of dimension $(n-1)$. This result was originally due to Farmer [Fa].

Exercise 72. (The central stability homology of $M(W_d)$.) Let W_d be an S_d–representation. Use the result of Exercise 71 to show that

$$\widetilde{H}_p(M(W_d))_n = 0 \qquad \text{for all } p \neq n-d-1.$$

We now define central stability, one notion of 'representation stability' for FI–modules, and document some of its consequences.

Definition XLII. (Central stability degree.) An FI–module V has *central stability degree* $\leq d$ if, for all $n > d$,

$$\widetilde{H}_{-1}(V)_n = \widetilde{H}_0(V)_n = 0.$$

An FI–module with finite central stability degree is called *centrally stable.*

One immediate consequence of central stability is that the tail of the central stability chain complex is exact in the stable range, and therefore gives a presentation for V_n in terms of V_{n-1} and V_{n-2} in this range. This and some additional consequences of central stability are collected in the following proposition. See (for example) Patzt [Pa] and the references therein for proofs.

Theorem XLIII. ***(Consequences of central stability.)*** *Suppose that an FI–module V has central stability degree $\leq d$. Then the following hold.*

- *For all $n > d$, V_n admits the following partial resolution (where the maps are described in Definition XL):*

$$\mathrm{Ind}_{S_{n-2}}^{S_n} V_{n-2} \longrightarrow \mathrm{Ind}_{S_{n-1}}^{S_n} V_{n-1} \longrightarrow V_n \longrightarrow 0$$

- *V is presented in degree $\leq d$ (compare to Church–Ellenberg [CE, Propositoin 4.2]).*
- *For $n > d$, $V_n = \mathrm{colim}_{\substack{S \subseteq [n] \\ |S| \leq d}} V_S$.*
- *V has polynomial degree $\leq d$ in FI–degree $> 2d-1$.*
- *The central stability homology groups vanish in a range:*

$$\widetilde{H}_p(V)_n = 0 \qquad \text{for all } n > 3d+p+1.$$

- *There exists a resolution of V by FI♯–modules P^i*

$$\cdots \longrightarrow P^i \longrightarrow P^{i-1} \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow V \longrightarrow 0$$

  where P^i is generated in degree $\leq 3d+i+1$.

Moreover, if V has presentation degree $\leq d$, then V is centrally stable in degree $\leq d+1$.

Possible additional topics: Kan extensions, local cohomology, local degree and stable degree, periodicity, connections to tca's

Michigan Representation Stability Week 2018
University of Michigan, Ann Arbor • 13–17 August 2018

An introduction to FI–modules and their generalizations

Jenny Wilson

Lecture 2: FI–modules in geometry, topology, and combinatorics

7 A warm-up case: The pure braid group

Recall that a *braid* on n strands can be visualized as an equivalence class of braid diagrams, as in Figure 4. A diagram represents n strands in Euclidean 3–space that are anchored at their startpoints at n distinguished points in a plane, and at the their endpoints at the same n points in a parallel plane. The strands may move in space but may not double back or pass through each other. These diagrams form a group under concatenation called the *braid group* $\mathbf{B}_n$.

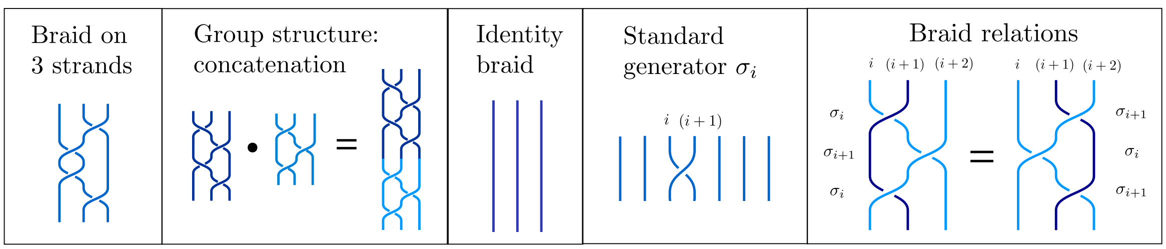


Figure 4: The braid group $\mathbf{B}_n$

Each braid defines a permutation on the endspoints of the n strands. The *pure braid group* $\mathbf{PB}_n$ is the kernel of the induced surjection

$$1 \longrightarrow \mathbf{PB}_n \longrightarrow \mathbf{B}_n \longrightarrow S_n \longrightarrow 1.$$

These are the braids where each strand begins and ends at the same point.

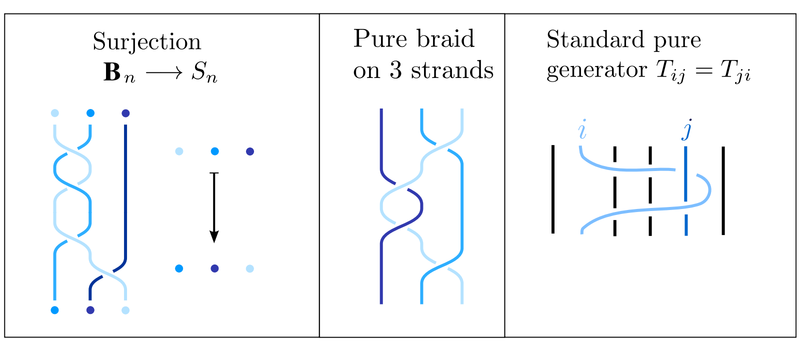


Figure 5: The pure braid group $\mathbf{PB}_n$

The pure braid group $\mathbf{PB}_n$ has a $(2n)$–real dimensional $K(\pi, 1)$ space,

$$F_n(\mathbb{C}) := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for all } i \neq j\}.$$

We take $\pi_1(F_n(\mathbb{C}))$ as the definition of the pure braid group $\mathbf{PB}_n$. We may interpret $F_n(\mathbb{C})$ as the *ordered configuration space of* $\mathbb{C}$, and view a point in $F_n(\mathbb{C})$ as an embedding of n labelled points in $\mathbb{C}$.

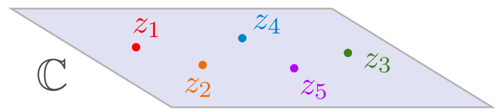


Figure 6: A point in $F_n(\mathbb{C})$

Alternatively, we may interpret $F_n(\mathbb{C})$ as the complement of the arrangement of hyperplanes $z_i = z_j$ in $\mathbb{C}^n$; these are the hyperplanes fixed by the reflections of the symmetric group S_n under its canonical action on $\mathbb{C}^n$. These different perspectives on $F_n(\mathbb{C})$ suggest different families of generalizations, which we will discuss below.

The symmetric group S_n acts on $F_n(\mathbb{C})$ freely by permuting the coordinates. The quotient space

$$C_n(\mathbb{C}) := F_n(\mathbb{C})/S_n$$

is a $K(\pi, 1)$ space for the braid group $\mathbf{B}_n$, and we take its fundamental group as the definition of the braid group.

7.1 The cohomology of the pure braid group

The following exercise implies that the (co)homology of the pure braid group on n strands is a representation of S_n.

Exercise 73. Let

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be a short exact sequence of groups. The group G acts on K by conjugation, and this action induces an action of G on the (co)homology groups $H^*(K; R)$ and $H_*(K; R)$. Explain why the action of G on (co)homology factors through an action of the quotient $Q \cong G/K$.
Hint: See Brown [Bro, Proposition 6.2 and Corollary 6.3].

The abelianization of the pure braid group is the free abelian group on the $\binom{n}{2}$ generators $T_{i,j}$.

$$H_1(\mathbf{PB}_n) = \mathbb{Z} \cdot \{T_{i,j}\} \cong \mathbb{Z}^{\binom{n}{2}}, \qquad H^1(\mathbf{PB}_n) = \operatorname{Hom}(\mathbf{PB}_n, \mathbb{Z}) = \mathbb{Z} \cdot \{T^*_{i,j}\} \cong \mathbb{Z}^{\binom{n}{2}}$$

We may interpret the class $T^*_{i,j}$ as measuring "winding number"; $T^*_{i,j}$ takes a pure braid and counts the number of times (with sign) that strand wraps around strand j. Viewing $F_n(\mathbb{C}) \subseteq \mathbb{C}^n$ as a complex manifold, we can identify the class $T^*_{i,j}$ with the meromorphic form

$$\omega_{i,j} := \frac{1}{2\pi I}\left(\frac{dz_i - dz_j}{z_i - z_j}\right), \qquad I \text{ a square root of } -1,$$

and again interpret $\omega_{i,j}$ as measuring the "winding number" of a loop around the deleted hyperplane $z_i = z_j$. It is not difficult to check by hand that these forms satisfy the identity

$$\omega_{i,j} \wedge \omega_{j,k} + \omega_{j,k} \wedge \omega_{k,i} + \omega_{k,i} \wedge \omega_{i,j} = 0 \qquad \text{for distinct } i, j, k \in [n]. \tag{1}$$

The action of S_n on $F_n(\mathbb{C})$ induces an action on these forms by

$$\sigma \cdot \omega_{i,j} = \omega_{\sigma(i),\sigma(j)} \qquad \text{for } \sigma \in S_n.$$

Arnold [A] proved that the integer cohomology ring $H^*(\mathbf{PB}_n) = H^*(F_n(\mathbb{C}))$ is generated as a ring by the degree-1 elements $\omega_{i,j}$. It is in fact exactly the $\mathbb{Z}$–subalgebra generated by these forms, and defined by the relation in Equation 1. Two exterior polynomials in these forms are cohomologous if and only if they are equal.

Theorem XLIV. ***(The cohomology of*** $F_n(\mathbb{C})$***.)****(Arnold [A, Theorem 1 & Corollary 3].)*
The cohomology algebra $H^*(F_n(\mathbb{C}))$ is the exterior graded algebra generated by the $\binom{n}{2}$ forms $\omega_{i,j}$, which are subject to the $\binom{n}{3}$ relations in Equation 1:

$$H^*(F_n(\mathbb{C})) \cong \frac{\bigwedge_{\mathbb{Z}}^* \omega_{i,j}}{\langle \omega_{q,r} \wedge \omega_{r,s} + \omega_{r,s} \wedge \omega_{s,q} + \omega_{s,q} \wedge \omega_{q,r} \rangle} \qquad \begin{array}{l} i,j,q,r,s \in [n], \\ i,j \text{ distinct}, \quad q,r,s \text{ distinct.} \end{array}$$

In degree p, the cohomology group $H^p(F_n(\mathbb{C}))$ is free abelian and has a basis of exterior monomials of the form

$$\omega_{i_1,j_1} \wedge \omega_{i_2,j_2} \wedge \cdots \wedge \omega_{i_p,j_p}, \qquad \text{where } i_s < j_s, \text{ and } j_1 < j_2 < \cdots < j_p.$$

7.2 $H^*(F_n(\mathbb{C})) = H^*(\mathbf{PB}_n)$ is a finitely generated FI–module

Exercise 74. **($H^p(\mathbf{PB}_n)$ as an FI–module.)**

(a) Show that there are group homomorphisms $\mathbf{PB}_n \to \mathbf{PB}_{n+1}$ and $\mathbf{PB}_{n+1} \to \mathbf{PB}_n$ defined by adding or deleting a strand.

(b) Describe the corresponding maps on the spaces $F_n(\mathbb{C})$.

(c) Show that, for each fixed p, the sequence of S_n–representations $\{H^p(\mathbf{PB}_n; \mathbb{Z})\}_n$ has the structure of an FI–module over $\mathbb{Z}$.

(d) Show that moreover $\{H^p(\mathbf{PB}_n; \mathbb{Z})\}_n$ has the structure of an FI♯–module.

(e) Fix p. Show that as an FI–module $\{H^p(\mathbf{PB}_n; \mathbb{Z})\}_n$ is finitely generated in degree $\leq 2p$.

By the structure theorem for FI♯–modules (Exercise 47), for each p we can express $\{H^p(\mathbf{PB}_n; \mathbb{Q})\}_n$ in the form $\bigoplus_{0 \leq d \leq 2p} M(W_d)$ for some representations W_d.

Exercise 75. **(The FI♯–module structure on $H^p(\mathbf{PB}_n)$.)**

(a) Verify that $\{H^1(\mathbf{PB}_n; \mathbb{Q})\}_n$ is the FI♯–module $M(V_{\square\square})$.

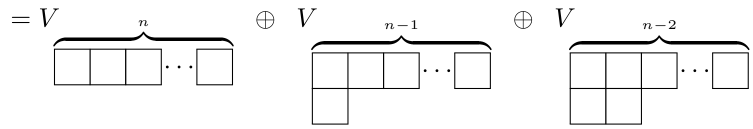
(b) Find the S_d–representations W_d such that $\{H^2(\mathbf{PB}_n; \mathbb{Q})\}_n$ is the FI♯–module $\bigoplus_{0 \leq d \leq 4} M(W_d)$.

By Exercise 54, it follows that the sequence $\{H^p(\mathbf{PB}_n; \mathbb{Q})\}_n$ is uniformly multiplicity stable with stable range $n \geq 4p$.

Exercise 76. (Multiplicity stability for $\{H^p(\mathbf{PB}_n;\mathbb{Q})\}_n$ for small p)

(a) Prove that, as an S_n–representation, the decomposition of $H^1(\mathbf{PB}_n;\mathbb{Q})$ into irreducible representations is

$$H^1(\mathbf{PB}_n;\mathbb{Q}) = V(0)_n \oplus V(1)_n \oplus V(2)_n$$



for all $n \geq 4$.

(b) Explicitly identify the subrepresentations

$$V(0)_n \cong \mathbb{Q} \qquad \text{and} \qquad V(0)_n \oplus V(1)_n \cong \mathbb{Q}^n$$

as vector subspaces of $H^1(\mathbf{PB}_n;\mathbb{Q}) = \text{span}_{\mathbb{Q}}(\omega_{i,j})$.

(c) Compute the decomposition of $H^2(\mathbf{PB}_n;\mathbb{Q})$ into irreducible representations. Verify that it is multiplicity stable for all $n \geq 8$, as predicted by Exercise 54.

Another consequence of Exercise 74 is that, by Exercise 58, the sequence $H^p(\mathbf{PB}_n;\mathbb{Q})$ has a character polynomial of degree $\leq 2p$.

Exercise 77. (Character polynomials for $H^p(\mathbf{PB}_n)$.) Let χ_n^p denote the character for $H^p(\mathbf{PB}_n)$.

(a) Verify that $\chi_n^1 = X_2 + \binom{X_1}{2}$ for all $n \geq 0$.

(b) Compute the characters χ_n^2 and verify that that they coincide with a character polynomial of degree ≤ 4.

7.3 A model proof of central stability, following Quillen

Since $H_q(\mathbf{PB}_n)$ is an $\mathsf{FI}\sharp$–module generated in degree $\leq 2q$, it follows that it has presentation degree $\leq 2q$ and central stability degree $\leq 2q+1$. However, in the interest of using this warm-up case to illustrate a technique for proving central stability, we will forget the $\mathsf{FI}\sharp$–module structure and these known bounds. Variations on this proof have been used in the literature, for example, to prove representation stability results for congruence subgroups of $\text{GL}_n(R)$, for the Torelli subgroup of the mapping class group, and for the Torelli subgroup of $\text{Aut}(F_n)$.

From the short exact sequences

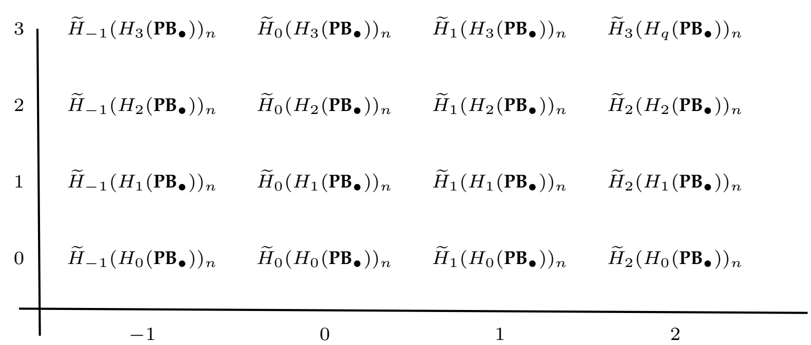
$$1 \longrightarrow \mathbf{PB}_n \longrightarrow \mathbf{B}_n \longrightarrow S_n \longrightarrow 1$$

we can construct a double complex for each n, and obtain two spectral sequences; see Putman–Sam [PS] for details. The E^1 page of the first spectral sequence comes from a certain simplicial complex associated to the braid group. Hatcher–Wahl [HW] determined that this simplicial complex is highly connected, as a special case of a homological stability proof for mapping class groups. It follows from their result that the limit $E^\infty_{p,q}$ vanishes for $n \geq p+q+2$. The second spectral sequence converges to the same limit.

The second spectral sequence has E^2 page

$$E^2_{p,q}(n) = \widetilde{H}_p\Big(H_q(\mathbf{PB}_\bullet;\mathbb{Z})\Big)_n$$

The q^{th} row of this E^2 page are the central stability homology groups of the FI–module $H_q(\mathbf{PB}_n;\mathbb{Z})$.

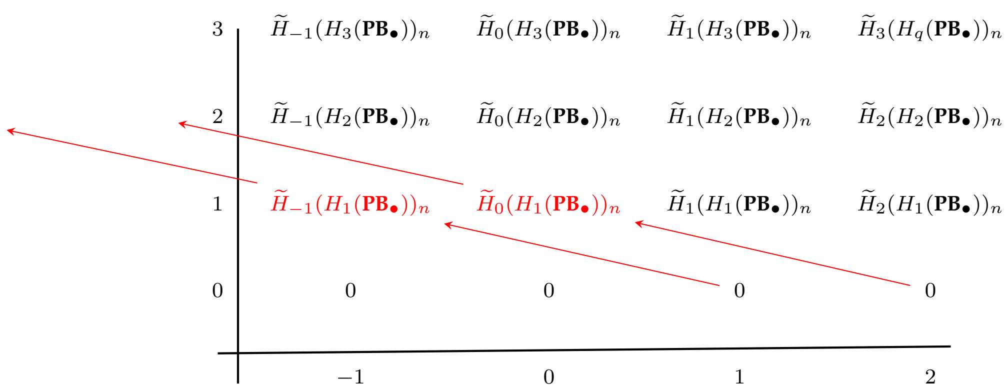


Exercise 78.

(a) Use the observation that $H_0(\mathbf{PB}_n) = \mathbb{Z}$ and Exercise 71 to find a range in n in which terms on the bottom row $\widetilde{H}_p(H_0(\mathbf{PB}_n))$ vanishes.

(b) Since the first four terms on the bottom $q = 0$ row vanish in a stable range, in this range there are no possible nonzero differentials to or from the groups

$$E^2_{-1,1} = \widetilde{H}_{-1}\Big(H_1(\mathbf{PB}_\bullet)\Big)_n \qquad \text{or} \qquad E^2_{0,1} = \widetilde{H}_0\Big(H_1(\mathbf{PB}_\bullet)\Big)_n.$$



Since the spectral sequence converges to zero in a range, these terms must eventually be zero. Find a range in which these terms vanish. This bounds the central stability degree of the FI–module $H_1(\mathbf{PB}_n)$.

(c) To propagate the argument for $q = 2$, we can use the fact that

$$\widetilde{H}_{-1}(V)_n = \widetilde{H}_0(V)_n = 0 \quad \text{for } n \gg 0 \quad \Longrightarrow \quad \widetilde{H}_p(V)_n = 0 \quad \text{for } n \gg p.$$

Use Theorem XLIII to find a range where the terms $\widetilde{H}_p(H_1(\mathbf{PB}_\bullet))_n$ vanish, and use this computation to bound the central stability degree of the FI–module $H_2(\mathbf{PB}_n)$.

(d) Use induction to find a bound on the central stability degree of the FI–module $H_q(\mathbf{PB}_n)$.

8 FI–modules in geometry and topology

In this section we will outline a number of applications of FI–module theory to objects in geometry and topology.

8.1 Hyperplane complements

Definition XLV. (The hyperplane arrangement of a reflection group.) Let G be a group of linear maps acting on $\mathbb{R}^d$, generated by a finite set of reflections. Let $\{s_i\}$ denote the set of all reflections in G, and let H_i denote the hyperplane fixed by s_i. The set of hyperplanes $\{H_i\}$ is called the *reflection arrangement* associated to G, and we define its complex *hyperplane complement*

$$\mathcal{M}_G := \mathbb{C}^d \backslash \{\text{union of complexified hyperplanes } H_i \otimes_{\mathbb{R}} \mathbb{C} \}$$

If $G \cong S_n$ is the group of $n \times n$ permutation matrices, then $\mathcal{M}_{S_n} = F_n(\mathbb{C})$ and its fundamental group is $\mathbf{PB}_n$. More generally, the fundamental group of the quotient $\mathcal{M}_G/G$ is called the *generalized braid group*, and the fundamental group of $\mathcal{M}_G$ is the *pure generalized braid group* associated to G.

Since G stabilizes the set of complex hyperplanes $\{H_i \otimes_{\mathbb{R}} \mathbb{C}\}$, it has a well-defined action on the complement $\mathcal{M}_G$. In particular, when $S_n \subseteq G$, the cohomology groups $H^*(\mathcal{M}_G)$ have an action of S_n. Many of the properties of the cohomology of $\mathcal{M}_{S_n}$ hold for general reflection arrangements, as we see with the following results of Brieskorn.

Theorem XLVI (Brieskorn [Bri, Théorème 6(i).]). ***(The cohomology of a hyperplane complement $\mathcal{M}_G$).*** *Let G be a finite reflection group. Then the cohomology groups $H^p(\mathcal{M}_G)$ of the complex hyperplane complement $\mathcal{M}_G$ are free abelian, with rank*

$$\textit{rank } H^p(\mathcal{M}_G) = \#\{ \ g \in G \mid \textit{length}(g) = p \ \}$$

where the length is taken with respect to the generating set of all reflections in G.

Theorem XLVII (Brieskorn [Bri, Lemme 5]). ***(Generating the cohomology of $\mathcal{M}$).*** *Let $\mathcal{M}$ be the complement of a finite arrangement of hyperplanes in a complex vector space V. Suppose each hyperplane H_i is determined by a linear form ℓ_i. Then the cohomology algebra of the complex hyperplane complement $\mathcal{M}$ is generated by the differential forms*

$$\omega_i := \frac{1}{2\pi I}\left(\frac{d\ell_i}{\ell_i}\right).$$

Moreover, the cohomology algebra is isomorphic to the $\mathbb{Z}$–subsalgebra of meromorphic forms on V generated by the forms ω_i.

Remark XLVIII. Orlik and Solomon [OS] proved that if $\mathcal{M}$ is the complement of a finite arrangement of complex hyperplanes, then the cohomology of $\mathcal{M}$ is completely determined by the combinatorial data of the poset of the hyperplanes' intersections (under inclusion). They give a presentation for the cohomology $H^*(\mathcal{M})$ as an algebra.

Exercise 79. (Representation stability for the cohomology of a hyperplane complement.) For each n, let $\mathcal{M}_n$ be the complement of a finite arrangement of hyperplanes in $\mathbb{C}^n$ defined by linear functionals $\{\ell_i^n\}$.

(a) Show that the natural action of FI♯ on the spaces $\{\mathbb{C}^n\}$ induces an action of $\text{FI}\sharp^{op} = \text{FI}\sharp$ on the space of linear functionals on $\mathbb{C}^n$.

(b) Suppose that the set of functionals $\sqcup_{n\geq 1}\{\ell_i^n\}$ are stable under the FI♯ action. Show that the cohomology groups $H^p(\mathcal{M}_n;\mathbb{Z})$ form an FI♯–module for each p.

(c) Suppose that set of functionals $\sqcup_{n\geq 1}\{\ell_i^n\}$ is finitely generated under the action of FI. Use Brieskorn's result Theorem XLVII to show that this FI–module is finitely generated.

The structure of the cohomology of hyperplane complements – and their generalizations – have been studied in a representation stability context (for example) by Church, Ellenberg, and Farb [CF, CEF1, CEF2], Jiménez Rolland and Wilson [Wi2, JW2], Bibby [Bi], and Gadish [Ga].

8.2 Configuration spaces

Definition XLIX. (Ordered and unordered configuration spaces.) For a topological space M, define the *ordered configuration space of M on n points* to be the space

$$F_n(M) = \{(m_1, \ldots, m_n) \in M^n \mid m_i \neq m_j \text{ for all } i \neq j\},$$

topologized as a subspace of M^n. Equivalently, $F_n(M)$ is the space of embeddings

$$[n] \hookrightarrow M,$$

and we can visualize elements of $F_3(M)$ as in Figure 7.

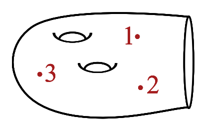


Figure 7: A point in $F_3(M)$.

The symmetric group S_n acts on $F_n(M)$ by permuting the coordinates. The quotient space $C_n(M)$ under this action is called the *unordered configuration space of M on n points*. This is the space

$$C_n(M) = \Big\{\{m_1, \ldots, m_n\} \subseteq M\Big\}$$

of n–point subsets of M.

The fundamental groups $\pi_1(C_n(M))$ and $\pi_1(F_n(M))$ are called the *braid group of M* and *pure braid group of M*, respectively, however, in contrast to the case $M = \mathbb{C}$, these configuration spaces need not be $K(\pi, 1)$ spaces in general.

8.2.1 The cohomology of configuration space as an FI–module

Fix a topological space M. Given a finite set S, write $F_S(M)$ to denote the space of embeddings $S \hookrightarrow M$, so $F_n(M) = F_{[n]}(M)$. Then the spaces $F_S(M)$ have the structure of a contravariant functor from FI: for each injective map $\alpha : S \hookrightarrow T$ there is a map $\alpha^* : F_T(M) \to F_S(M)$ obtained by precomposing an embedding $T \hookrightarrow M$ with α. This operation is shown in Figure 8.

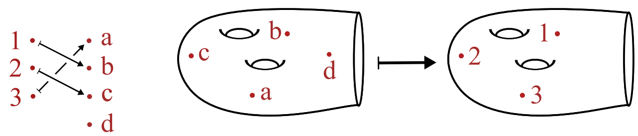


Figure 8: The contravariant action of an FI morphism on configuration space

It follows that the homology groups $\{H_*(F_S(M)) \mid S \text{ finite}\}$ form an FI^{op}–module, and the cohomology groups $\{H^*(F_S(M)) \mid S \text{ finite}\}$ form an FI–module.

If M is an open manifold, then it is also possible to define maps $F_n(M) \to F_{n+1}(M)$ by rescaling a configuration and introducing a new point labelled $(n+1)$ "at infinity". For open manifolds of dimension at least 2, then, there is an action of FI (well-defined up to homotopy) on the configuration spaces $F_S(M)$, as illustrated in Figure 9. Thus when M is open, the homology and cohomology groups of the spaces $F_S(M)$ have FI♯–module structures.

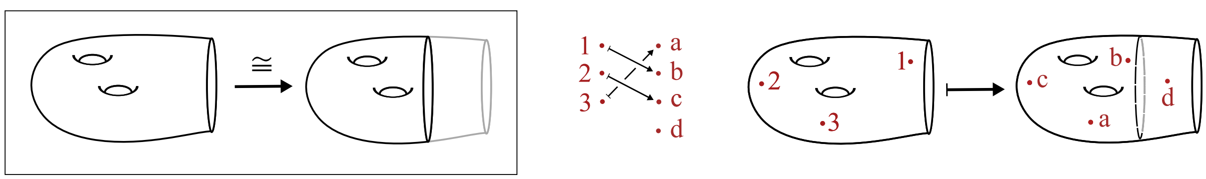


Figure 9: The action of an FI morphism on the configuration space of an open manifold

There is an extensive literature on the cohomology of the configuration spaces of manifolds, though the cohomology groups are known explicitly in very few instances. In a representation stability context, there are results on these cohomology groups due to Church [Ch, Theorem 1], Church–Ellenberg–Farb [CEF1, Section 6], Church–Ellenberg–Farb–Nagpal [CEFN, Application 2], Hersh–Reiner [HR], Church–Miller–Nagpal–Reinhold [CMNR, Section 4], Bahran [Ba, Theorem 1] and others.

Theorem L. ***(Representation stability for the cohomology of configuration spaces of a manifold.)*** *(Church–Miller–Nagpal–Reinhold [CMNR, Application A].)*
Suppose that M is a connected manifold of dimension at least 2. Then for fixed q, the FI*–module $\{H^q(F_n(M))\}_n$ has generation degree $\leq \max(0, 10q-1)$ and presentation degree $\leq \max(0, 18q-2)$.*

Other representation stability results on configuration spaces include the following. Kupers–Miller [KM] proved stability results for the homotopy groups of configuration spaces of manifolds. Ellenberg–Wiltshire-Gordon [EW] exhibited additional algebraic structure on the cohomology of configuration spaces for manifolds with nowhere vanishing vector fields. Miller–Wilson established patterns called *secondary representation stability* in the ccohomology of configuration spaces of open manifolds. Tosteson [To1], Petersen [Pe], Ramos [Ra1, Ra2], and Lütgehetmann [Lü] have studied the configuration spaces of spaces such as graphs or finite CW–complexes.

8.3 Mapping class groups and moduli spaces of surfaces with marked points

Definition LI. (The mapping class group Mod(Σ).) Let Σ be an orientable smooth surface. Recall that the *mapping class group* Mod(Σ) of Σ is the group of orientation-preserving diffeomorphisms that fix the boundary $\partial\Sigma$ pointwise, up to smooth isotopy fixing the boundary pointwise,

$$\text{Mod}(\Sigma) := \text{ Diffeo}^+(\Sigma, \partial\Sigma) \;/\; (\text{isotopy fixing } \partial\Sigma).$$

Equivalently, Mod(Σ) is the group of path components of the topological group Diffeo$^+(\Sigma, \partial\Sigma)$.

Example LII. (Dehn twists.) An important class of mapping classes are the *Dehn twists*. Given an embedded loop $\gamma \in \Sigma$, there is a neighbourhood of γ that is homeomorphic to an annulus. The Dehn twist T_γ is executed by twisting this annulus as in Figure 10, and extending by the identity to the complement of the annulus in Σ.

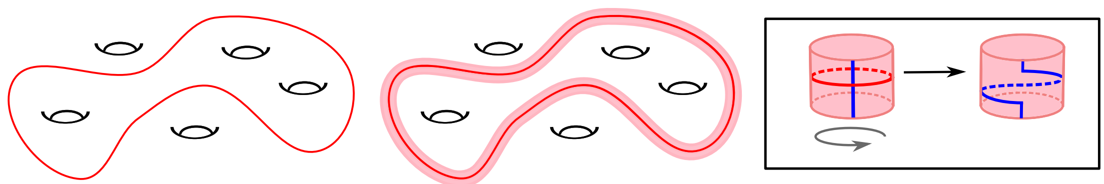


Figure 10: Performing the Dehn twist T_γ in an annular neighbourhood of γ

It follows from work of Dehn, Mumford, Lickorish, Humphries, and others that the mapping class group of a compact genus g surface with 0 or 1 boundary components is finitely generated by $(2g+1)$ Dehn twists.

Definition LIII. (The pure mapping class group Mod(Σ).) Given a set of n labelled marked points in a surface Σ, denote by Diffeo$^{n,+}(\Sigma, \partial\Sigma)$ the group of orientation preserving diffeomorphism that fix $\partial\Sigma$ and the marked points pointwise. The *pure mapping class group* is the group

$$P\text{Mod}(\Sigma) := \text{ Diffeo}^{n,+}(\Sigma, \partial\Sigma) \;/\; (\text{isotopy fixing } \partial\Sigma \text{ and the } n \text{ labelled points}).$$

We use the shorthand $P\text{Mod}^n_{g,b}$ for when $\Sigma = \Sigma^n_{g,b}$ is a compact genus-g surface with b boundary components.

Assume that $\Sigma^n_{g,b}$ is a surface not homeomorphic to S^2, $\mathbb{R}^2$, the closed disk, the torus, the closed annulus, the once-punctured disk, or the once-punctured plane. The *(generalized) Birman exact sequence* relates the pure mapping class group of $\Sigma^n_{g,b}$ to its surface braid group and the mapping class group of the surface $\Sigma_{g,b} = \Sigma^0_{g,b}$ without marked points.

$$1 \longrightarrow \pi_1(F_n(\Sigma_{g,b})) \longrightarrow P\text{Mod}^n_{g,b} \longrightarrow \text{Mod}(\Sigma_{g,b}) \longrightarrow 1.$$

Definition LIV. (The moduli space of Riemann surfaces.) We write $\mathcal{M}_{g,n}$ to denote the *moduli space of Riemann surfaces* of genus g with n marked points. An element $(X, p) \in \mathcal{M}_{g,n}$ is an equivalence class, up to biholomorphism, of a Riemann surface X of genus g and a set of labelled marked points $p \in F_n(X)$.

For $g \geq 2$, the moduli space $\mathcal{M}_{g,n}$ is a rational model for the classifying space $BP\text{Mod}^n_{g,0}$, hence

$$H^*(\mathcal{M}_{g,n}; \mathbb{Q}) = H^*(P\text{Mod}^n_{g,0}; \mathbb{Q}).$$

These cohomology groups are a topic of significant interest in low-dimensional topology and algebraic geometry. Jiménez Rolland proved the following.

Theorem LV. ***(Representation stability for the mapping class groups and moduli spaces of surfaces with marked points.)*** *(Jiménez Rolland [JR, Theorems 1.1, 6.1, 6.3].)*
Let R be a field. For any $q \geq 0$ and $2g + b > 2$ the FI–module $H_q(PMod^{\bullet}_{g,r}; R)$ is finitely generated over R. If $R = \mathbb{Q}$ then the characters are given by a character polynomial of degree $\leq 2q$, its terms have weight $\leq 2q$, and the sequence is multiplicity stable with stable range $n \geq 6q$. For $r > 0$ it is multiplicity stable for $n \geq 4q$.

Jiménez Rolland and Maya Duque [JD, Theorems 1.1 and 1.2] later proved a representation stability result for the rational cohomology of the real locus $\overline{\mathcal{M}_{0,n}}(\mathbb{R})$ of the Deligne–Mumford compactification $\overline{\mathcal{M}_{0,n}}$ of the moduli space of rational curves with n marked points. For fixed genus $g > 0$, however, the cohomology groups $H^i(\overline{\mathcal{M}_{g,n}}; \mathbb{Q})$ can grow exponentially in n and therefore these sequence may be infinitely generated as an FI–modules. Tosteson [To2, Theorem 1.2] proved, however, that the sequences of rational homology groups $\{H_i(\overline{\mathcal{M}_{g,n}}; \mathbb{Q})\}_n$ are subquotients of finitely generated FS^{op}–modules, where FS^{op} is the opposite category of the category of finite sets and surjective maps. Tosteson [To2, Theorem 1.1] deduces a polynomial growth condition and constraints on the irreducible S_n–subrepresentations that occur.

8.4 Congruence subgroups of $\textbf{GL}_n(K)$

The congruence subgroups of $\text{GL}_n(\mathbb{Z})$ are certain families of finite index subgroups that play a role in topology and number theory.

Definition LVI. (Level I principal congruence subgroups.) Let K be a commutative ring and $I \subseteq K$ a proper ideal. Then the *level I congruence subgroups* $\text{GL}_n(K, I)$ of $\text{GL}_n(K)$ are defined to be the kernel of the "reduction modulo I" maps

$$1 \longrightarrow \text{GL}_n(K, I) \longrightarrow \text{GL}_n(K) \longrightarrow \text{GL}_n(K/I).$$

Of particular interest are the *level ℓ congruence subgroups* $\text{GL}_n(\mathbb{Z}, \ell\mathbb{Z})$ of $\text{GL}_n(\mathbb{Z})$.

Exercise 80. Show that the permutation matrices S_n are contained in the image of $\text{GL}_n(K)$ in $\text{GL}_n(K/I)$. Conclude from Exercise 73 that the homology groups $H_p(\text{GL}_n(K, I))$ are S_n–representations.

Exercise 81. Use the inclusions

$$\begin{aligned} \text{GL}_n(K) &\longrightarrow \text{GL}_{n+1}(K) \\ [A] &\longmapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

to define an FI–module structure on the sequence of homology groups $\{H_p(\text{GL}_n(K, I))\}_n$.

The following result is due to Gan–Li [GL], and is the culmination of a literature on representation stability results for congruence subgroups due to Putman [Pu], Church–Ellenberg–Farb–Nagpal [CEFN], Church–Ellenberg [CE], Church–Miller–Nagpal–Reinhold [CMNR], and others.

Theorem LVII. ***(Representation stability for congruence subgroups.)*** *(Gan–Li [GL, Theorems 1 and 11].) Suppose that K is a ring satisfying Bass's stable range condition SR_{d+2} for some $d > 0$ and that I is a proper two-sided ideal of K. Then the FI–module $H_q(GL_n(K, I); \mathbb{Z})$ has generation degree $\leq 4q + 2d + 1$ and presentation degree $\leq 4q + 2d + 6$. Moreover, for each $q > 0$ and $n > 0$ there is a canonical isomorphism:*

$$\mathit{colim}_{S \subseteq [n], S \leq \omega(q)} H_q(GL_S(K, I); \mathbb{Z}) \xrightarrow{\cong} H_q(GL_n(K, I); \mathbb{Z}) \qquad \text{where } \omega(q) = 4q + 2d + 6.$$

8.5 Flag varieties and coinvariant algebras

8.5.1 The cohomology of the complete flag variety

Definition LVIII. (The complete flag varietry.) Let $\mathbf{B}_n \subseteq \mathrm{GL}_n(\mathbb{C})$ be the subgroup of upper triangular matrices. The space $\mathrm{GL}_n(\mathbb{C})/\mathbf{B}_n$ is the *complete flag variety*

$$\mathrm{GL}_n(\mathbb{C})/\mathbf{B}_n \cong \{0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq \mathbb{C}^n\}, \qquad V_i \text{ subspaces.}$$

Borel proved that the cohomology algebra $H^{2*}(\mathrm{GL}_n(\mathbb{C})/\mathbf{B}_n; \mathbb{C})$ is isomorphic to an algebra called the complex coinvariant algebra.

Definition LIX. (The complex coinvariant algebra.) The complex *coinvariant algebra* $\mathcal{C}^*(n)$ is a graded algebra defined as the quotient

$$\mathcal{C}^*(n) = \mathbb{C}[x_1, \ldots, x_n]/I_n$$

where I_n is the homogeneous ideal generated by the S_n–invariant polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ with constant term equal to zero.

Theorem LX. ***(The cohomology algebra $H^{2*}(GL_n(\mathbb{C})/B_n; \mathbb{C})$.)*** *(Borel [Bo]). There is an isomorphism of graded algebras*

$$H^{2*}(GL_n(\mathbb{C})/B_n; \mathbb{C}) \cong \mathcal{C}^*(n).$$

> **Exercise 82. ($\{\mathcal{C}^*(n)\}$ as an FI^{op}–module.)** Show that the homogeneous graded pieces of the coinvariant algebras $\mathcal{C}^*(n)$ are stable under the action of FI^{op} induced by its action on $\mathbb{C}[x_1, \ldots, x_n]$, but that the action of FI does not induce a well-defined FI–module structure on $\mathcal{C}^*(n)$.
>
> **Exercise 83. (Representation stability for the graded pieces of $\{\mathcal{C}^*(n)\}$ in low degree.)** Explicitly compute as S_n–representations the homogeneous degree-k graded pieces $\mathcal{C}^k(n)$ in degree $k = 0, 1, 2$. Verify that, for each k, the sequence is multiplicity stable and that the characters agree with a character polynomial for all n sufficiently large.

The following result is originally due Church–Farb [CF, Theorem 7.4] and Church–Ellenberg–Farb [CEF1, Proposition 5.1. and Theorem 1.11].

> **Exercise 84. (Representation stability for the graded pieces $\{\mathcal{C}^k(n)\}$.)** Fix k, and consider the homogeneous degree-k graded pieces $\{\mathcal{C}^k(n)\}$ of the coinvariant algebras. Show that this sequence of S_n–representations is uniformly multiplicity stable, and that its characters eventually agree with a character polynomial. *Hint:* You can obtain an FI–module from an FI^{op}–module by dualizing. See [CEF1, Sections 4.2 and 5.1].

8.5.2 Diagonal coinvariant algebras

The following generalizations of the coinvariant algebras play a role in algebraic combinatorics.

Definition LXI. (The r–diagonal coinvariant algebra.) Consider the polynomial algebra

$$P_n^r = \mathbb{C}[x_1^{(1)}, \ldots, x_n^{(1)}, x_1^{(2)}, \ldots, x_n^{(2)}, \ldots, x_1^{(r)}, \ldots, x_n^{(r)}]$$

in r sets of variables, with a diagonal action of the symmetric groups S_n. The r–diagonal coinvariant algebra $\mathcal{C}(r,n)$ is defined as the quotient

$$\mathcal{C}(r,n) = P_n^r / I_n^r$$

where I_n^r is the homogeneous ideal generated by the constant-term-zero S_n–invariant polynomials.

The following result is due to Church–Ellenberg–Farb [CEF1, Proposition 5.1. and Theorem 1.11].

> **Exercise 85. (Representation stability for the multigraded pieces of $\{\mathcal{C}(r,n)\}$.)** Fix r, and fix a multigrading $(k_1, k_2, \ldots, k_r)$. Consider the homogeneous degree-$(k_1, k_2, \ldots, k_r)$ graded pieces of the diagonal coinvariant algebras $\{\mathcal{C}(r,n)\}_n$. Show that this sequence of S_n–representations is uniformly multiplicity stable, and that its characters eventually agree with a character polynomial.

8.5.3 Generalized flag varieties

Let $\mathbf{G}_n^{\mathcal{W}}$ be a semisimple complex Lie group in type A_{n-1}, B_n, C_n, or D_n, with Weyl group $\mathcal{W}_n$ and $\mathbf{B}_n^{\mathcal{W}}$ a Borel subgroup. Then the space $\mathbf{G}_n^{\mathcal{W}}/\mathbf{B}_n^{\mathcal{W}}$ is called a *generalized flag variety*. Borel [Bo] showed that its cohomology algebra is isomorphic to the associated coinvariant algebra, defined as the quotient of $\mathbb{C}[x_1, x_2, \ldots x_n]$ by the constant-term-zero $\mathcal{W}_n$–invariant polynomials. In each case $S_n \subseteq \mathcal{W}_n$, and the cohomology algebras form a FI^{op}–module. These FI^{op}–module structures have been studied by, for example, Church, Ellenberg, and Farb [CF, CEF1, CEF2], and Wilson, Jiménez Rolland, and Fulman [Wi1, JW1, FJW].

Michigan Representation Stability Week 2018
University of Michigan, Ann Arbor • 13–17 August 2018

An introduction to FI–modules and their generalizations

Jenny Wilson

Lecture 3: Representation stability and linear categories

9 Representation stability and linear groups

There is a growing literature on generalizations of FI–modules designed to address sequences of S_n–representations with additional structure, and to address sequences of representations of other families of groups. In this section we will focus on the work of Putman–Sam [PS] and others on representations of classical linear groups. Some applications of their work, including representation stability results for Torelli groups and congruence subgroups, are outlined in section 10.

9.1 The categories

To develop an analogue of FI–modules that encodes sequences of G_n–representations for a family of linear groups $\{G_n\}$, we seek a category whose endomorphisms are isomorphic to the groups G_n. For each category $\mathcal{C}$, we call a functors $\mathcal{C} \to R$–Mod *$\mathcal{C}$–modules* and study the category $\mathcal{C}$–Mod of $\mathcal{C}$–modules and natural transformations. The concepts of presentation degree (Definition XXXVI), polynomial degree (Definition XXXIX) may all be adapted to the categories, and central stability degree (Definition XLII) makes sense for modules over $\mathsf{VIC}(A)$, $\mathsf{VIC}^H(A)$, and $\mathsf{SI}(A)$.

Definition LXII. (The category $\mathsf{VI}(A)$.) Let A be a commutative ring. Let $\mathsf{VI}(A)$ be the category whose objects are free A–modules of finite rank and whose morphisms are injective linear maps with left inverses.

Exercise 86. Show that the full subcategory on objects A^n, $n \geq 0$, is a skeleton for $\mathsf{VI}(A)$.

Exercise 87.

(a) Show that the assignments $V \mapsto \bigwedge^k V,\ V \mapsto \mathrm{Sym}^k(V)$, and $V \mapsto \otimes^k V$ all define $\mathsf{VI}(A)$–modules.

(b) Show that these $\mathsf{VI}(A)$–modules are polynomial of degree k.

The following variation on VI was introduced by Djament [Dj1].

Definition LXIII. (The category $\mathsf{VIC}(A)$.) Let A be a commutative ring. Let $\mathsf{VIC}(A)$ be the category whose objects are free A–modules of finite rank and whose morphisms are given by an injective linear map together with a choice of direct complement of the image. Concretely,

$$\mathrm{Hom}_{\mathsf{VIC}(A)}(V,W) = \left\{ (f,C) \;\middle|\; \begin{array}{l} f\colon V \hookrightarrow W \text{ an injective } A\text{–linear map,} \\ C \subseteq W \text{ a free submodule with } \mathrm{im} f \oplus C = W \end{array} \right\}.$$

The composition law is defined by

$$(f,C) \circ (g,D) = (f \circ g, C \oplus f(D)).$$

Exercise 88. Show that the full subcategory on objects A^n, $n \geq 0$, is a skeleton for $\mathsf{VIC}(A)$.

Exercise 89. Show that there is a surjection $\mathsf{VIC}(A) \to \mathsf{VI}(A)$, but that this surjection does not split.

Exercise 90.

(a) Does the sequence of groups $\{\mathrm{GL}_n(A)\}$ and their embeddings have the structure of a functor from $\mathsf{VI}(A)$ to the category of groups?
(b) Is it a functor from $\mathsf{VIC}(A)$ to the category of groups?

Exercise 91. Complete the following computations in when $\mathcal{C}$ is $\mathsf{VI}(A)$ and when it is $\mathsf{VIC}(A)$.

(a) Show that $\mathrm{End}_{\mathcal{C}}(A^n) \cong \mathrm{GL}_n(A)$ acts transitively on $\mathrm{Hom}_{\mathcal{C}}(A^m, A^n)$ by postcomposition.
(b) What is the stabilizer of the inclusion $A^m \hookrightarrow A^n$?
(c) Fix d. Describe the representable functor $R \cdot \mathrm{Hom}_{\mathcal{C}}(A^d, -)$.
(d) Suppose that $A = \mathbb{F}_q$ is a finite field. Now how does the rank of the R–modules in the sequence
$$n \longmapsto R \cdot \mathrm{Hom}_{\mathcal{C}}(A^d, A^n)$$
grow with n?
(e) Let $A = \mathbb{F}_q$. Deduce that the representable functor $R \cdot \mathrm{Hom}_{\mathcal{C}}(A^d, -)$ does not have finite polynomial degree. Contrast this result with Exercise 69.

Nagpal [Na2] proved that finitely generated $\mathsf{VI}(\mathbb{F}_q)$–modules do satisfy a polynomial growth condition. This result was proved by Gan–Watterlond [GW1, Theorem 1.7] over $\mathbb{Q}$.

Theorem LXIV. ***(q-polynomiality of dimension of $\mathsf{VI}(\mathbb{F}_q)$–modules.)*** *(Nagpal [Na2, Theorem 1.1].) Assume that R is a field in which q is invertible. Let M be a finitely generated $\mathsf{VI}(\mathbb{F}_q)$–module. Then there exists a polynomial P such that $\dim_R M(\mathbb{F}_q^n) = P(q^n)$ for all n sufficiently large.*

The following variation on $\mathsf{VIC}(A)$ allows for actions of general linear groups with restricted determinant.

Definition LXV. (The category $\mathsf{VIC}^H(A)$.) Let R be a commutative ring and H a subgroup of the group of units $A^\times$. Let $\mathrm{GL}_n^H(A)$ denote the subgroup of $\mathrm{GL}_n(A)$ given by
$$\mathrm{GL}_n^H(R) = \{B \in \mathrm{GL}_n(A) \mid \det(B) \in H\}.$$
For a nonzero finite-rank free A–module V, define an *H–orientation* on V to be a generator of
$$\bigwedge^{\mathrm{rank}(V)} V \cong A$$
considered up to multiplication by H. We now define the category $\mathsf{VIC}^H(A)$ as follows. Its objects are finite-rank free A–modules V such that nonzero objects are assigned an *H–orientation*. If $\mathrm{rank}(V) = \mathrm{rank}(W)$, then the morphisms $\mathrm{Hom}_{\mathsf{VIC}^H(A)}(V, W)$ are linear isomorphisms that respect the chosen H–orientations. In particular, the endomorphisms satsify
$$\mathrm{Hom}_{\mathsf{VIC}^H(A)}(V, V) = \mathrm{GL}^H(V).$$
For $\mathrm{rank}(V) \neq \mathrm{rank}(W)$, the endomorphisms
$$\mathrm{Hom}_{\mathsf{VIC}^H(A)}(V, W) \cong \mathrm{Hom}_{\mathsf{VIC}(A)}(V, W)$$
are again injective complemented linear maps (f, C), and we assign to C the (unique) H–orientation that makes the H–orientations on $(\mathrm{im} f \oplus C)$ and W agree. See Putman–Sam [PS, Section 1.2].

Definition LXVI. (The category $\mathsf{SI}(R)$.) Let A be a commutative ring and let $\mathsf{SI}(A)$ be the category of free finite-rank symplectic A–modules and isometric embeddings. Details are given in Putman–Sam [PS, Section 1.2].

One of the main results of Putman–Sam [PS] is the local Noetherian property for these categories in the case when A is a finite ring. Their proof uses a category theoretic notion of Gröbner bases, a variant on the theory developed by Sam–Snowden [SS3].

Theorem LXVII. ***(The local Noetherian property.)*** *(Putman–Sam [PS, Theorems A, B, C, D, E].) Let A be a finite commutative ring. Let R be a Noetherian commutative ring. Then the categories of $\mathsf{VI}(A)$–, $\mathsf{VIC}(A)$–, $\mathsf{VIC}^H(A)$–, and $\mathsf{SI}(A)$–modules over R are locally Noetherian.*

Exercise 92. (Failure of Noetherianity for A infinite.) (Putman–Sam [PS, Theorem N].)

(a) Suppose that A contains $\mathbb{Z}$. Show that $\mathrm{SL}_2(A)$ contains a free group of rank 2.
(b) Suppose that A is an infinite ring of finite characteristic. Show that A is not finitely additively generated. Then, find an injective map from the additive group of A to $\mathrm{SL}_2(A)$.
(c) Suppose that A is an infinite ring. Show that $\mathsf{VI}(A)$ and $\mathsf{VIC}(A)$ are not locally Noetherian.

Analogues of multiplicity stability (Definition XXVII) are known for finitely generated modules over these categories in several cases. See Gan–Watterlond [GW1, Theorem 1.6] for $\mathsf{VI}(\mathbb{F}_q)$–modules, Gan–Watterlond [GW2, Theorem 4] for $\mathsf{VIC}(\mathbb{F}_q)$–modules, and Patzt [Pa, Theorem A and B] for rational $\mathsf{VIC}(\mathbb{Q})$– and rational $\mathsf{SI}(\mathbb{Q})$–modules.

Although we no longer have all the equivalences of finite presentation degree that hold for FI–modules (see Theorem XLIII), finite presentation degree does imply the following.

Theorem LXVIII. ***(Consequences of finite presentation degree.)***

- *(Djament [Dj2, Proposition 2.14]; see also Patzt [Pa, Proposition 6.1] and Putman–Sam [PS, Theorem F].) Let $\mathcal{C}$ be one of the above categories, and let M be a $\mathcal{C}$–module presented in degree $\leq d$. For an object $V \in \mathcal{C}$ let $|V| = \frac{1}{2}rank(V)$ if $\mathcal{C} = \mathsf{SI}(A)$ and $|V| = rank(V)$ otherwise. Then*

$$M_U = colim_{V \in \mathcal{C}, |V| \leq d} M_V \qquad \text{for all } |V| \geq d.$$

- *(Patzt [Pa, Corollary 6.4(b)].) Let A be a ring with stable rank s. Let M be a $\mathsf{VIC}(A)$–module generated in degree $\leq d$. Then M is presented in degree $\leq (d+s+1)$ if and only if it is centrally stable in degree $\leq (d+s+1)$.*
- *(Patzt [Pa, Corollary 6.4(c)].) Let A be a ring with unitary stable rank s. Let M be an $\mathsf{SI}(A)$–module generated in degree $\leq d$. Then M is presented in degree $\leq (d+s+2)$ if and only if it is centrally stable in degree $\leq (d+s+2)$.*

10 Applications to geometry and topology

10.1 Torelli groups

Let $\Sigma_{g,b}$ denote a compact orientable smooth genus g surface with b boundary components. For $b = 0, 1$, recall that the first homology of $\Sigma_{g,b}$ is the free abelian group

$$H_1(\Sigma_{g,b}) \cong \mathbb{Z}^{2g}$$

with basis shown in Figure 11, and recall that the intersection pairing on these curves endows $H_1(\Sigma_{g,b})$ with a symplectic structure.

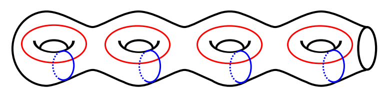


Figure 11: A $\mathbb{Z}$–basis for $H_1(\Sigma_{g,1})$

Recall the definition of the mapping class group of $\Sigma_{g,b}$ from subsection 8.3. The induced action of the mapping class group on $H_1(\Sigma_{g,b})$ respects the intersection pairing, and hence for $b = 0, 1$ defines a representation $\mathrm{Mod}(\Sigma_{g,b}) \to \mathrm{Sp}_{2g}(\mathbb{Z})$. It is well-known that this representation is surjective.

Exercise 93. (Mod$(T^2) =$ Sp$_2(\mathbb{Z})$.) Let α and β denote the curves on the closed torus $T^2 = \Sigma_{1,0}$ shown in Figure 12. The mapping class group $\mathrm{Mod}(T^2)$ is generated by the Dehn twists T_α and T_β (Theorem LII). Show that their action on $H_1(T^2)$ is represented by the following matrices:

$$\begin{aligned} \mathrm{Mod}(T^2) &\longrightarrow \mathrm{Sp}_2(\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z}) \\ T_\alpha &\longmapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ T_\beta &\longmapsto \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

This representation defines an isomorphism $\mathrm{Mod}(T^2) \cong \mathrm{Sp}_2(\mathbb{Z})$.

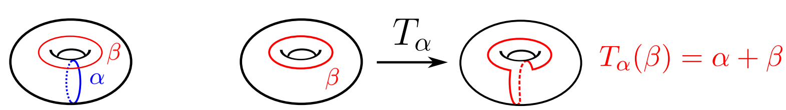


Figure 12: The action of the Dehn twist T_α on the homology class of β

Unlike in genus $g = 1$, in general this symplectic representation has a large kernel.

Definition LXIX. The Torelli groups For $b = 0, 1$, the *Torelli subgroup* $\mathcal{I}_{g,b}$ of the mapping class group of $\Sigma_{g,b}$ is the kernel of the symplectic representation

$$1 \longrightarrow \mathcal{I}_{g,b} \longrightarrow \mathrm{Mod}(\Sigma_{g,b}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

The subgroups $\mathcal{I}_{g,1}$ are generated by the elements shown in Figure 13.

Johnson computed the abelianization of $\mathcal{I}_{g,b}$ in a series of papers in the 1980s. For $p > 1$, the groups $H_p(\mathcal{I}_{g,b})$ remain largely mysterious. It is unknown for most p and g, for example, whether these groups are finitely generated. There is some hope, however, that we may be able to understand these homology groups in the framework of representation stability.

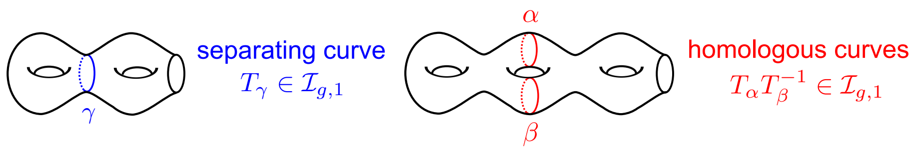


Figure 13: Some elements in the Torelli group $\mathcal{I}_{g,1}$

10.1.1 The homology of the Torelli groups as $\mathsf{SI}(\mathbb{Z})$–modules

We now specialize to the case of surfaces with $b = 1$ boundary component. By Exercise 73, there is an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $H_p(\mathcal{I}_{g,1})$. Moreover, the embeddings $\Sigma_{g,1} \hookrightarrow \Sigma_{g+1,1}$ allow us to promote diffeomorphisms of $\Sigma_{g,1}$ to $\Sigma_{g+1,1}$ by extending by the identity. The resultant maps $\mathrm{Mod}(\Sigma_{g,1}) \to \mathrm{Mod}(\Sigma_{g+1,1})$ respect the Torelli subgroup, and we therefore obtain $\mathrm{Sp}_{2g}(\mathbb{Z})$–equivariant maps

$$H_p(\mathcal{I}_{g,1}) \longrightarrow H_p(\mathcal{I}_{g+1,1}).$$

Exercise 94. Show for each $p \geq 0$ that the sequence of homology groups $\{H_p(\mathcal{I}_{g,1})\}_g$ has the structure of an $\mathsf{SI}(\mathbb{Z})$–module.

Open Problem 95. Show that, for each $p \geq 0$, the $\mathsf{SI}(\mathbb{Z})$–module $\{H_p(\mathcal{I}_{g,1})\}_g$ is presented in finite degree. Find bounds on the degrees of the generators and relators.

The result of Problem 95 is trivial when $p = 0$, and it follows from Johnson's description of the groups $H_1(\mathcal{I}_{g,1})$ when $p = 1$. For $p = 2$, the problem was solved by Miller–Patzt–Wilson [MPW, Theorem B].

Theorem LXX. *(Miller–Patzt–Wilson [MPW, Theorem B and 3.9].) The $\mathsf{SI}(\mathbb{Z})$–module $H_2(\mathcal{I}_g)$ has central stability degree ≤ 45. It is generated in degree ≤ 21 and presented in degree ≤ 45.*

More recently, Kassabov–Putman [KP, Theorem A] proved that $\{H_2(\mathcal{I}_{g,1})\}_g$ is finitely generated as an $\mathsf{SI}(\mathbb{Z})$–module. Problem 95 is open for $g \geq 3$.

10.2 IA_n and $\mathrm{Aut}(F_n)$

Definition LXXI. (The Torelli subgroup of $\mathrm{Aut}(F_n)$.) Let F_n denote the free group on n letters. Given an automorphism $F_n \to F_n$, there is an induced automorphism on the abelianization $\mathbb{Z}^n$ of F_n. This construction defines a homomorphism $\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$. It is not difficult to verify that this homomorphism is surjective, and its kernel IA_n is a group of interest in geometric group theory.

Exercise 96. (The homology of IA_n as $\mathsf{VIC}(\mathbb{Z})$–modules.)

(a) Use the short exact sequence

$$1 \longrightarrow \mathrm{IA}_n \longrightarrow \mathrm{Aut}(F_n) \longrightarrow \mathrm{GL}_n(\mathbb{Z}) \longrightarrow 1$$

to show that $H_*(\mathrm{IA}_n)$ is a $\mathrm{GL}_n(\mathbb{Z})$–representation.

(b) Show that, for each q, the sequence $\{H_q(\mathrm{IA}_n)\}_n$ has a $\mathsf{VIC}(\mathbb{Z})$–module structure.

A result of Day and Putman [DP, Theorem B] implies that $H_2(\mathrm{IA}_n)$ is finitely generated as a $\mathsf{VIC}(\mathbb{Z})$–module in degree ≤ 6. Miller–Patzt–Wilson [MPW] proved a central stability result for these degree-2 homology groups.

Theorem LXXII. *(Miller–Patzt–Wilson [MPW, Theorem A and 3.9], Day and Putman [DP, Theorem B].) The* $\mathsf{VIC}(\mathbb{Z})$*–module* $H_2(IA_n)$ *has central stability degree* ≤ 38*. It is generated in degree* ≤ 6 *and presented in degree* ≤ 38.

Open Problem 97. Show that, for each $q \geq 3$, the $\mathsf{VIC}(\mathbb{Z})$–module $\{H_q(\mathrm{IA}_n)\}_n$ is presented in finite degree. Find bounds on the degrees of the generators and relators.

10.3 Congruence subgroups of $\mathbf{GL}_n(K)$, revisited

Recall Definition LVI, the definition of the level I congruence subgroup $\mathrm{GL}_n(K, I)$ of $\mathrm{GL}_n(K)$.

Exercise 98. (A $\mathsf{VIC}^H(K/I)$–module structure on $H_q(\mathbf{GL}_n(K;I))$.) Let K be a commutative ring and I a proper ideal.

(a) Recall the defining exact sequence

$$1 \longrightarrow \mathrm{GL}_n(K, I) \longrightarrow \mathrm{GL}_n(K) \longrightarrow \mathrm{GL}_n(K/I).$$

Define the group H to be the image of $K^\times$ in K/I, and recall that $\mathrm{GL}_n^H(K/I)$ is the subgroup of $\mathrm{GL}_n(K/I)$ of matrices with determinant in H. Show that $\mathrm{GL}_n^H(K/I)$ is the image of $\mathrm{GL}_n(K)$ in $\mathrm{GL}_n(K/I)$.

(b) Deduce that the sequence of homology groups $\{H_q(\mathrm{GL}_n(K;I))\}_n$ has the structure of a $\mathsf{VIC}^H(K/I)$–module.

Theorem LVII showed that the sequence $\{H_q(\mathrm{GL}_n(K;I))\}_n$ is finitely presented as an FI–module. Exercise 98 raises the question of whether this sequence stabilizes as a $\mathsf{VIC}^H(K/I)$–module. Miller–Patzt–Wilson [MPW, Theorem C], buliding on Putman–Sam [PS, Section 1.5], proved the following.

Theorem LXXIII. ***(Central stability for $H_q(\mathbf{GL}_n(K;I))$ as a $\mathsf{VIC}^H(K/I)$–module.)***
(Miller–Patzt–Wilson [MPW, Theorem C]. See also Putman–Sam [PS, Theorem G]).
Let I be a proper ideal of a commutative ring K. Let t be the minimal stable rank of all rings containing the ideal I, and assume that K/I is a PID of stable rank s. Then the sequence $H_q(GL_n(K, I))$ has central stability degree

$$\begin{aligned} &\leq s+1 && \text{for } q = 0 \\ &\leq \max(5+t, 5+s) && \text{for } q = 1 \\ &\leq (2q-1)(6t+21) - 10 + s && \text{for } q \geq 2 \end{aligned}$$

as $GL_n^H(K/I)$–representations.

References

[A] Arnold, Vladimir I. "The cohomology ring of the colored braid group." Vladimir I. Arnold-Collected Works. Springer, Berlin, Heidelberg, 1969. 183–186.

[Ba] Bahran, Cihan. "An improvement in the linear stable ranges for ordered configuration spaces." arXiv preprint arXiv:1801.03302 (2018).

[Bi] Bibby, Christin. "Representation stability for the cohomology of arrangements associated to root systems." Journal of Algebraic Combinatorics (2017): 1–25. arXiv:1603.08131.

[Bo] Borel, Armand. "Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts." Annals of Mathematics (1953): 115-207.

[Bri] Brieskorn, Egbert. "Sur les groupes de tresses [d'après VI Arnol'd]." Sminaire Bourbaki vol. 1971/72 Exposs 400417. Springer, Berlin, Heidelberg, 1973. 21–44.

[Bro] Brown, Kenneth S. *Cohomology of groups.* Vol. 87. Springer Science & Business Media, 2012.

[Ch] Church, Thomas. "Homological stability for configuration spaces of manifolds." Inventiones mathematicae 188.2 (2012): 465-504. arXiv:1103.2441.

[CF] Church, Thomas, and Benson Farb. "Representation theory and homological stability." Advances in Mathematics 245 (2013): 250–314. arXiv:1008.1368.

[CE] Church, Thomas, and Jordan Ellenberg. "Homology of FI–modules." Geometry & Topology 21.4 (2017): 2373-2418. arXiv:1506.01022.

[CEF1] Church, Thomas, Jordan S. Ellenberg, and Benson Farb. "FI–modules and stability for representations of symmetric groups." Duke Mathematical Journal 164.9 (2015): 1833–1910. arXiv:1204.4533.

[CEFN] Church, T., Ellenberg, J. S., Farb, B., & Nagpal, R. "FI–modules over Noetherian rings." Geometry & Topology 18.5 (2014): 2951–2984. arXiv:1210.1854.

[CEF2] Church, Thomas, Jordan Ellenberg, and Benson Farb. "Representation stability in cohomology and asymptotics for families of varieties over finite fields." Contemporary Mathematics 620 (2014): 1–54. arXiv:1309.6038.

[CMNR] Church, T., Miller, J., Nagpal, R., and Reinhold, J. "Linear and quadratic ranges in representation stability." Advances in Mathematics 333 (2018): 1–40. arXiv:1706.03845.

[DP] Day, Matthew, and Andrew Putman. "On the second homology group of the Torelli subgroup of $\mathrm{Aut}(F_n)$." Geometry & Topology 21.5 (2017): 2851-2896. arXiv:1408.6242.

[Dj2] Djament, Aurélien. "Des propriétés de finitude des foncteurs polynomiaux". Fundamenta Mathematicae 233 (2016): 197–256.

[Dj1] Djament, Aurélien. "Sur l'homologie des groupes unitaires coefficients polynomiaux." Journal of K-theory 10.1 (2012): 87–139.arXiv:1104.1507.

[DV] Djament, Aurélien, and Christine Vespa. "Weakly polynomial functors." arXiv preprint arXiv:1308.4106. (2013).

[EW] Ellenberg, Jordan S., and John D. Wiltshire-Gordon. "Algebraic structures on cohomology of configuration spaces of manifolds with flows." arXiv preprint arXiv:1508.02430 (2015).

[Fa] Farmer, Frank D. "Cellular homology for posets." Math. Japon 23.6 (1978): 79.

[FJW] Fulman, Jason, Rita Jiménez Rolland, and Jennifer CH Wilson. "Generating functions and statistics on spaces of maximal tori in classical Lie groups." New York J. Math 23 (2017): 165–191. arXiv:1610.06816.

[Ga] Gadish, Nir. "Representation stability for families of linear subspace arrangements." Advances in Mathematics 322 (2017): 341–377. arXiv:1603.08547.

[GL] Gan, Wee Liang, and Liping Li. "Linear stable range for homology of congruence subgroups via FI–modules." arXiv:1712.00890 (2017).

[GW1] Gan, Wee Liang, and John Watterlond. "A representation stability theorem for VI–modules." Algebras and Representation Theory 21.1 (2018): 47-60. arXiv:1602.00654.

[GW2] Gan, Wee Liang, and John Watterlond. "Stable decompositions of certain representations of the finite general linear groups." Transformation Groups (2018): 1-11. arXiv:1605.08434.

[HR] Hersh, Patricia, and Victor Reiner. "Representation Stability for Cohomology of Configuration Spaces in $\mathbb{R}^d$." International Mathematics Research Notices 2017.5 (2016): 1433–1486. arXiv:1505.04196.

[HW] Hatcher, Allen, and Nathalie Wahl. "Stabilization for mapping class groups of 3–manifolds." Duke Mathematical Journal 155.2 (2010): 205–269. arXiv:0709.2173.

[JR] Jiménez Rolland, Rita. "On the cohomology of pure mapping class groups as FI–modules." Journal of Homotopy and Related Structures 10.3 (2015): 401-424. arXiv:1207.6828.

[JD] Jiménez Rolland, Rita, and Joaqun Maya Duque. "Representation stability for the pure cactus group." Contributions of Mexican Mathematicians Abroad in Pure and Applied Mathematics 709 (2018): 53. arXiv:1501.02835.

[JW1] Jiménez Rolland, Rita, and Jennifer CH Wilson. "Convergence criteria for $\mathrm{FI}_{\mathcal{W}}$–algebras and polynomial statistics on maximal tori in type B/C." Advances in Mathematics 310 (2017): 121–158. arXiv:1604.06392.

[JW2] Jiménez Rolland, Rita, and Jennifer CH Wilson. "Stability for hyperplane complements of type B/C and statistics on squarefree polynomials over finite fields." arXiv preprint arXiv:1703.06349 (2017).

[KM] Kupers, Alexander, and Jeremy Miller. "Representation stability for homotopy groups of configuration spaces." Journal fr die reine und angewandte Mathematik (Crelles Journal) 2018.737 (2018): 217-253. arXiv:1410.2328.

[KP] Kassabov, Martin, and Andrew Putman. "Equivariant group presentations and the second homology group of the Torelli group." arXiv preprint arXiv:1807.01338 (2018).

[L] Littlewood, D. E. "Products and plethysms of characters with orthogonal, symplectic and symmetric groups." Canad. J. Math 10 (1958): 17–32.

[Lü] Lütgehetmann, Daniel. "Representation Stability for Configuration Spaces of Graphs." arXiv preprint arXiv:1701.03490 (2017).

[Ma] Macdonald, Ian Grant. *Symmetric functions and Hall polynomials.* Oxford university press, 1998.

[MPW] Miller, Jeremy, Peter Patzt, and Jennifer CH Wilson. "Central stability for the homology of congruence subgroups and the second homology of Torelli groups." arXiv preprint arXiv:1704.04449 (2017).

[MW] Miller, Jeremy, and Jennifer CH Wilson. "Higher order representation stability and ordered configuration spaces of manifolds." arXiv preprint arXiv:1611.01920 (2016).

[Mu] Murnaghan, Francis D. "On the characters of the symmetric group." Proceedings of the National Academy of Sciences 41.6 (1955): 396–398.

[Na1] Nagpal, Rohit. "FI–modules and the cohomology of modular representations of symmetric groups." arXiv preprint arXiv:1505.04294 (2015).

[Na2] Nagpal, Rohit. "VI–modules in non-describing characteristic, Part I." arXiv preprint arXiv:1709.07591.

[OS] Orlik, Peter, and Louis Solomon. "Combinatorics and topology of complements of hyperplanes." Inventiones mathematicae 56.2 (1980): 167–189.

[Pa] Patzt, Peter. "Representation stability for filtrations of Torelli groups." Mathematische Annalen (2017): 1–42. arXiv:1608.06507.

[Pa] Patzt, Peter. "Central stability homology." arXiv preprint arXiv:1704.04128 (2017).

[Pe] Petersen, Dan. "A spectral sequence for stratified spaces and configuration spaces of points." Geometry & Topology 21.4 (2017): 2527–2555. arXiv:1603.01137.

[Pi] Pirashvili, Teimuraz. "Dold–Kan type theorem for Γ–groups." Mathematische Annalen 318.2 (2000): 277–298.

[Pu] Putman, Andrew. "Stability in the homology of congruence subgroups." Inventiones mathematicae 202.3 (2015): 987-1027. arXiv:1201.4876.

[PS] Putman, Andrew, and Steven V. Sam. "Representation stability and finite linear groups." Duke Mathematical Journal 166.13 (2017): 2521-2598. arXiv:1408.3694.

[Ra1] Ramos, Eric. "Stability phenomena in the homology of tree braid groups." Algebraic & Geometric Topology 18.4 (2018): 2305-2337.

[Ra2] Ramos, Eric. "An application of the theory of FI–algebras to graph configuration spaces." arXiv preprint arXiv:1805.05316 (2018).

[RWW] Randal-Williams, Oscar, and Nathalie Wahl. "Homological stability for automorphism groups." Advances in Mathematics 318 (2017): 534–626. arXiv:1409.3541.

[Sp] Specht, Wilhelm. "Die charaktere der symmetrischen gruppe." Mathematische Zeitschrift 73.4 (1960): 312–329.

[SS1] Sam, Steven V., and Andrew Snowden. "Introduction to twisted commutative algebras." arXiv preprint arXiv:1209.5122 (2012).

[SS2] Sam, Steven, and Andrew Snowden. "GL–equivariant modules over polynomial rings in infinitely many variables." Transactions of the American Mathematical Society 368.2 (2016): 1097–1158. arXiv:1206.2233.

[SS3] Sam, Steven, and Andrew Snowden. "Gröbner methods for representations of combinatorial categories." Journal of the American Mathematical Society 30.1 (2017): 159–203. arXiv:1409.1670.

[Sn] Snowden, Andrew. "Syzygies of Segre embeddings and Δ–modules." Duke Mathematical Journal 162.2 (2013): 225–277. arXiv:1006.5248.

[To1] Tosteson, Philip. "Lattice spectral sequences and cohomology of configuration spaces." arXiv preprint arXiv:1612.06034 (2016).

[To2] Tosteson, Philip. "Stability in the homology of Deligne–Mumford compactifications." arXiv preprint arXiv:1801.03894(2018).

[We] Weibel, Charles A. *An introduction to homological algebra.* No. 38. Cambridge university press, 1995.

[Wi1] Wilson, Jennifer CH. "$\mathsf{FI}_{\mathcal{W}}$–modules and stability criteria for representations of classical Weyl groups." Journal of Algebra 420 (2014): 269–332. arXiv:1309.3817.

[Wi2] Wilson, Jennifer CH. "$\mathsf{FI}_{\mathcal{W}}$–modules and constraints on classical Weyl group characters." Mathematische Zeitschrift 281.1-2 (2015): 1–42. arXiv:1503.08510.

[Wi3] Wilson, Jennifer. "A brief introduction to representation stability." *Topology of Arrangements and Representation Stability.* Mathematisches Forschungsinstitut Oberwolfach Report No. 2/2018 DOI: 10.4171. (2018). Preliminary version.

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