$\mathrm{FI}_{\mathcal{W}}$–modules and constraints on classical Weyl group characters

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Abstract

In this paper we study the characters of sequences of representations of any of the three families of classical Weyl groups $\mathcal{W}_n$: the symmetric groups, the signed permutation groups (hyperoctahedral groups), or the even-signed permutation groups. Our results extend work of Church, Ellenberg, Farb, and Nagpal [CEF12], [CEFN14] on the symmetric groups. We use the concept of an *$FI_{\mathcal{W}}$–module*, an algebraic object that encodes the data of a sequence of $\mathcal{W}_n$–representations with maps between them, defined in the author's recent work [Wil14].

We show that if a sequence $\{V_n\}$ of $\mathcal{W}_n$–representations has the structure of a *finitely generated $FI_{\mathcal{W}}$–module*, then there are substantial constraints on the growth of the sequence and the structure of the characters: for n large, the dimension of V_n is equal to a polynomial in n, and the characters of V_n are given by a *character polynomial* in signed-cycle-counting class functions, independent of n. We determine bounds the degrees of these polynomials.

We continue to develop the theory of $\mathrm{FI}_{\mathcal{W}}$–modules, and we apply this theory to obtain new results about a number of sequences associated to the classical Weyl groups: the cohomology of complements of classical Coxeter hyperplane arrangements, and the cohomology of the pure string motion groups (the groups of symmetric automorphisms of the free group).

Contents

1 Introduction

In recent work [Wil14] we developed the theory of *$FI_{\mathcal{W}}$–modules* to study sequences of representations of any of the three families of classical Weyl groups: the symmetric groups S_n, the hyperoctahedral groups (signed permutation groups) $B_n \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$, and their index-two subgroups of even-signed permutation groups D_n. The results generalize work of Church, Ellenberg, Farb, and Nagpal [CEFN14, CEF12] on sequences of S_n–representations.

Let $\mathcal{W}_n$ denote one of these families. Many naturally occurring sequences of $\mathcal{W}_n$–representations $\{V_n\}_n$ with maps $V_n \to V_{n+1}$ carry the structure of an *finitely generated $FI_{\mathcal{W}}$–module*, an elementary and often easily established condition which we define in detail below.

We analyze two such families of sequences in Section 5. The first is the cohomology of the pure string motion groups $P\Sigma_n$, groups related to the pure braid groups. The group $P\Sigma_n$ can also be identified with the group of pure symmetric automorphisms of the free group F_n. The second family is the cohomology of the (complexified) complements $\mathscr{M}_{\mathcal{W}}(n)$ of the hyperplanes fixed by reflections in the Coxeter groups $\mathcal{W}_n$. The hyperplane complements $\mathscr{M}_{\mathcal{W}}(n)$ are objects of classical and current mathematical interest.

In this paper we develop general results on the characters of finitely generated $\mathrm{FI}_{\mathcal{W}}$–modules over characteristic zero. We prove that these characters admit very specific descriptions: they are (for n large) given by *character polynomials*, polynomials in signed-cycle-counting class functions, as described in Section 4.

Theorem 4.6. (Finitely generated $\mathrm{FI}_{\mathcal{W}}$–modules have character polynomials). *Let V be an $FI_{\mathcal{W}}$–module over characteristic zero finitely generated in degree $\leq d$. Let χ_{V_n} denote the character of the $\mathcal{W}_n$–representation V_n. Then there exists a unique character polynomial F_V of degree at most d such that $F_V(\sigma) = \chi_{V_n}(\sigma)$ for all $\sigma \in \mathcal{W}_n$, for all n sufficiently large.*

Church–Ellenberg–Farb proved this result in type A [CEF12, Theorem 2.67]. Theorem 4.16 gives a more detailed statement of the result in type B/C and D, including bounds on the stable range.

These character polynomials provide a description of the characters of V_n that is independent of n, and moreover the character polynomials' restrictive structure reflects the strong constraints on the $\mathcal{W}_n$–subrepresentations that can appear in a finitely generated $\mathrm{FI}_{\mathcal{W}}$–module.

One immediate consequence of the existence of character polynomials is that the dimensions $\dim(V_n)$ are (for n large) equal to a polynomial in n. We prove in Theorem 4.20 that this same phenomenon holds for sequences of representations $\{V_n\}_n$ over arbitrary fields when V_n admits a finitely generated $\mathrm{FI}_{\mathcal{W}}$–module structure. Since $\mathcal{W}_n$–representations over positive characteristic need not be completely reducible, the study of these sequences is more subtle – but Theorem 4.20 nonetheless offers some control over their growth.

Theorem 4.20. (Polynomial growth of dimension over arbitrary fields). *Let k be any field, and let V be a finitely generated $FI_{\mathcal{W}}$–module over k. Then there exists an integer-valued polynomial $P(T) \in \mathbb{Q}[T]$ such that*

$$dim_k(V_n) = P(n) \qquad \text{for all } n \text{ sufficiently large.}$$

Theorem 4.20 is proved in type A by Church–Ellenberg–Farb–Nagpal [CEFN14, Theorem 1.2].

These stability results for $\mathrm{FI}_{\mathcal{W}}$–modules extends our earlier work [Wil14]. We proved that if an $\mathrm{FI}_{\mathcal{W}}$–module over characteristic zero is finitely generated, the decompositions of the representations V_n into irreducible subrepresentations are in some sense eventually constant in n: these sequences are *uniformly representation stable* as defined by Church–Farb [CF13]. See Section 2.2 for a complete definition.

In this paper we introduce $\mathrm{FI}_{\mathcal{W}}\sharp$–modules, a subclass of $\mathrm{FI}_{\mathcal{W}}$–modules with additional symmetries that allow us to deduce even stronger restrictions on the structure of the underlying sequence of representations; see Section 3.

Examples of sequences of $\mathcal{W}_n$–representations with finitely generated $\mathrm{FI}_{\mathcal{W}}$–module and $\mathrm{FI}_{\mathcal{W}}\sharp$–module structures are prevalent throughout the fields of geometry, topology, algebra, and combinatorics. We establish this structure in two examples: the cohomology of the pure string motion groups $P\Sigma_n$ (Section 5.1) and the cohomology of the hyperplane complements $\mathscr{M}_{\mathcal{W}}(n)$ (Section 5.2). Theorem 1.1 summarizes our results.

Theorem 1.1. *Let $\mathcal{W}_n$ denote S_n, B_n, or D_n. Let X_n denote either the pure string motion groups $P\Sigma_n$ or the (complexified) complements $\mathscr{M}_{\mathcal{W}}(n)$ of the reflecting hyperplanes of $\mathcal{W}_n$. Then in each degree m the sequence $\{H^m(X_n;\mathbb{Q})\}_n$ is an $\mathrm{FI}_{\mathcal{W}}$–module finitely generated in degree $\leq 2m$. We show*

- *The sequence $H^m(X_n;\mathbb{Q})$ is uniformly representation stable, stabilizing for $n \geq 4m$.*
- *For all values of n, the characters of $H^m(X_n;\mathbb{Q})$ are given by a unique character polynomial of degree at most $2m$.*

These and other results for $P\Sigma_n$ and $\mathscr{M}_{\mathcal{W}}(n)$ are described in detail in Section 5.

The theory of $\mathrm{FI}_{\mathcal{W}}$–modules derives much of its strength from the extra algebraic structure provided to these sequences of $\mathcal{W}_n$–representations. In this framework each sequence $\{V_n\}_n$ is encoded as a single object, an $\mathrm{FI}_{\mathcal{W}}$–module, in a category that closely parallels the category of modules over a ring. In Section 2.3 we review the structure of maps, presentations, direct sums, and tensor products of $\mathrm{FI}_{\mathcal{W}}$–modules. We proved that $\mathrm{FI}_{\mathcal{W}}$–modules over Noetherian rings are Noetherian [Wil14, Section 4.3]; see Section 2.3.2. We developed induction and restriction operations between the three families of groups [Wil14, Sections 3.5 and 3.6]. These operations are reviewed in Section 2.3.4.

In our previous work [Wil14, Section 5.2] we gave an application analogous to Murnaghan's classical theorem for Kronecker coefficients [Mur38], a stability theorem concerning tensor products of S_n–representations. We showed that type B/C and D versions of Murnaghan's result follow readily from the $\mathrm{FI}_{\mathcal{W}}$–module theory [Wil14, Theorem 5.3 and Corollary 5.4]. These stability results have simple interpretations in this $\mathrm{FI}_{\mathcal{W}}$–module context: tensor products of finitely generated $\mathrm{FI}_{\mathcal{W}}$–modules are themselves finitely generated.

The theory of $\mathrm{FI}_{\mathcal{W}}$–modules gives a conceptual foundation and a language to describe stability phenomena in sequences of $\mathcal{W}_n$–representations such as these.

1.1 $\mathrm{FI}_{\mathcal{W}}$–modules and finite generation

Definition 1.2. (The Category $\mathrm{FI}_{\mathcal{W}}$). Here we will define the three categories $\mathrm{FI}_A \subseteq \mathrm{FI}_D \subseteq \mathrm{FI}_{BC}$, denoted generically by $\mathrm{FI}_{\mathcal{W}}$. Consider the category whose objects are $\varnothing$ and for each $n \in \mathbb{N}$ the finite set $\mathbf{n} = \{1, -1, 2, -2, \ldots, n, -n\}$, and whose morphisms are all injective maps. We define $\mathrm{FI}_{\mathcal{W}}$ to be the smallest subcategory containing the morphisms $\mathcal{W}_n \subseteq \mathrm{End}(\mathbf{n})$ and the canonical inclusion maps $I_n : \mathbf{n} \hookrightarrow (\mathbf{n+1})$.

In each case, the endomorphisms $\text{End}(\mathbf{n})$ of $\text{FI}_{\mathcal{W}}$ are precisely the Weyl group $\mathcal{W}_n$. The category FI_A is equivalent to the category of all finite sets and injective maps, denoted FI by [CEF12]. It turns out that for $n \neq m$, the set of morphisms $\mathbf{m} \to \mathbf{n}$ in FI_D and FI_{BC} are the same; see [Wil14, Remark 3.1].

A description of each category is given in Table 1.

Category	Objects	Morphisms
FI_{BC}	$\mathbf{n} = \{\pm 1, \pm 2, \ldots, \pm n\}$ $\mathbf{0} = \varnothing$	{ injections $f : \mathbf{m} \to \mathbf{n} \mid f(-a) = -f(a) \;\; \forall\, a \in \mathbf{m}$} $\text{End}(\mathbf{n}) \cong B_n$
FI_D	$\mathbf{n} = \{\pm 1, \pm 2, \ldots, \pm n\}$ $\mathbf{0} = \varnothing$	{ injections $f : \mathbf{m} \to \mathbf{n} \mid f(-a) = -f(a) \;\; \forall\, a \in \mathbf{m}$; isomorphisms must reverse an even number of signs } $\text{End}(\mathbf{n}) \cong D_n$
FI_A	$\mathbf{n} = \{\pm 1, \pm 2, \ldots, \pm n\}$ $\mathbf{0} = \varnothing$	{ injections $f : \mathbf{m} \to \mathbf{n} \mid f(-a) = -f(a) \;\; \forall\, a \in \mathbf{m}$; f preserves signs} $\text{End}(\mathbf{n}) \cong S_n$

Table 1: The Categories $\text{FI}_{\mathcal{W}}$

For $m < n$ we denote by $I_{m,n}$ the canonical inclusion $\{\pm 1, \ldots, \pm m\} \hookrightarrow \{\pm 1, \ldots, \pm n\}$ and abbreviate $I_n := I_{n,(n+1)}$.

Definition 1.3. ($\text{FI}_{\mathcal{W}}$–module and $\text{FI}_{\mathcal{W}}$–module maps). An *$\text{FI}_{\mathcal{W}}$–module* V over a commutative ring k is a (covariant) functor

$$V : \text{FI}_{\mathcal{W}} \longrightarrow k\text{–Mod}$$

from $\text{FI}_{\mathcal{W}}$ to the category of k–modules. We denote $V_n := V(\mathbf{n})$ and $f_* := V(f)$.

FI_A–modules are precisely the FI–modules studied in [CEF12, CEFN14].

A *co–$\text{FI}_{\mathcal{W}}$–module* over k is a functor from the dual category

$$\text{FI}_{\mathcal{W}}^{\text{op}} \longrightarrow k\text{–Mod}.$$

A *map of $\text{FI}_{\mathcal{W}}$–modules* $F : V \longrightarrow W$ is a natural transformation, that is, a sequence of maps $F_n : V_n \to W_n$ that commute with the action of the $\text{FI}_{\mathcal{W}}$ morphisms. $\text{FI}_{\mathcal{W}}$–module injections, quotients, kernels, cokernels, direct sums, etc, are defined pointwise.

Definition 1.4. (Finite generation, Degree of generation). Let V be an $\mathrm{FI}_{\mathcal{W}}$–module. Given a subset $S \subseteq \coprod_{n=0}^{\infty} V_n$, the sub–$\mathrm{FI}_{\mathcal{W}}$–module *generated by* S is the smallest sub–$\mathrm{FI}_{\mathcal{W}}$–module U of V containing S. S is a *generating set* for U: the images of these elements under the $\mathrm{FI}_{\mathcal{W}}$ morphisms span each $k[\mathcal{W}_n]$–module U_n.

An $\mathrm{FI}_{\mathcal{W}}$–module V is *finitely generated* if it has a finite generating set, and V is generated in *degree* $\leq d$ if it has a generating set contained in $\coprod_{n=0}^{d} V_n$.

Examples of $\mathrm{FI}_{\mathcal{W}}$–modules. To illustrate this concept we give some first examples of $\mathrm{FI}_{\mathcal{W}}$–modules over $\mathbb{Q}$. Fix the Weyl group family $\mathcal{W}_n$ to be S_n, D_n, or B_n. To specify the $\mathrm{FI}_{\mathcal{W}}$–module structure on a sequence $\{V_n\}$, it suffices to state the $\mathcal{W}_n$–actions and the maps $(I_n)_* : V_n \to V_{n+1}$. associated to the natural inclusions I_n.

Example 1.5. The following are $\mathrm{FI}_{\mathcal{W}}$–modules.

1. **Example: Trivial representations.** For $n \geq 0$ let $V_n = \mathbb{Q}$ be the trivial $\mathcal{W}_n$–representation with isomorphisms $(I_n)_* : V_n \cong V_{n+1}$. These spaces form an $\mathrm{FI}_{\mathcal{W}}$–module with a single generator in degree 0.

2. **Example: Signed permutation matrices.** The groups D_n and B_n are canonically represented by $n \times n$ *signed permutation matrices*, that is, generalized permutation matrices with nonzero entries equal to 1 or -1. Throughout Example 1.5 we let $\mathbb{Q}^n$ denote the representation of S_n, D_n, or B_n by (signed) permutation matrices. The representations $V_0 = 0$ and $V_n = \mathbb{Q}^n$ with their natural inclusions form an $\mathrm{FI}_{\mathcal{W}}$–module finitely generated in degree 1.

3. **Example: j-fold powers.** For any integer j, the j-fold tensor power, exterior power, and symmetric power on $\mathbb{Q}^n$ each form an $\mathrm{FI}_{\mathcal{W}}$–module finitely generated in degree j. More generally, composing any $\mathrm{FI}_{\mathcal{W}}$–module V with another functor k–Mod $\to$ k–Mod will yield a new $\mathrm{FI}_{\mathcal{W}}$–module.

4. **Example: Represented functors $M_{\mathcal{W}}(\mathbf{m})$.** For fixed integer $m \geq 0$, the sequence of k-modules
$$M_{\mathcal{W}}(\mathbf{m})_n := k\big[\mathrm{Hom}_{\mathrm{FI}_{\mathcal{W}}}(\mathbf{m}, \mathbf{n})\big]$$
form an $\mathrm{FI}_{\mathcal{W}}$-module, with $\mathrm{FI}_{\mathcal{W}}$ morphisms acting on basis elements $e_f, f \in \mathrm{Hom}_{\mathrm{FI}_{\mathcal{W}}}(\mathbf{m}, \mathbf{n})$ by postcomposition. These are in a sense the "free" $\mathrm{FI}_{\mathcal{W}}$–modules; see Definition 2.3.

5. **Example: The $\mathrm{FI}_{\mathcal{W}}$–modules $M_{\mathcal{W}}(U)$.** Fix an integer d and a $\mathcal{W}_d$–representation U. Let $\mathbb{Q}$ denote the trivial $\mathcal{W}_{n-d}$–representation. Let $U \boxtimes \mathbb{Q}$ denote the $(\mathcal{W}_d \times$

$\mathcal{W}_{n-d}$)–representation given by the *external tensor product* of U and $\mathbb{Q}$. Define

$$M_{\mathcal{W}}(W)_n := \begin{cases} 0, \ n < d \\ \mathrm{Ind}_{\mathcal{W}_d \times \mathcal{W}_{n-d}}^{\mathcal{W}_n} U \boxtimes \mathbb{Q}, \ n \geq d \end{cases}$$

Then there are induced maps $M_{\mathcal{W}}(U)_n \to M_{\mathcal{W}}(U)_{n+1}$ giving this sequence the structure of an $\mathrm{FI}_{\mathcal{W}}$–module finitely generated in degree d. Specifically, the natural inclusion of $(\mathcal{W}_d \times \mathcal{W}_{n-d})$–representations

$$\begin{aligned} U \boxtimes \mathbb{Q} &\hookrightarrow \mathrm{Res}_{\mathcal{W}_d \times \mathcal{W}_{n-d}}^{\mathcal{W}_{n+1}} \mathrm{Ind}_{\mathcal{W}_d \times \mathcal{W}_{n+1-d}}^{\mathcal{W}_{n+1}} U \boxtimes \mathbb{Q} \\ &= \mathrm{Res}_{\mathcal{W}_d \times \mathcal{W}_{n-d}}^{\mathcal{W}_n} \mathrm{Res}_{\mathcal{W}_n}^{\mathcal{W}_{n+1}} \mathrm{Ind}_{\mathcal{W}_d \times \mathcal{W}_{n+1-d}}^{\mathcal{W}_{n+1}} U \boxtimes \mathbb{Q} \end{aligned}$$

and the universal property of induction give $\mathcal{W}_n$-equivariant maps

$$\underbrace{\mathrm{Ind}_{\mathcal{W}_d \times \mathcal{W}_{n-d}}^{\mathcal{W}_n} U \boxtimes \mathbb{Q}}_{M_{\mathcal{W}}(U)_n} \longrightarrow \mathrm{Res}_{\mathcal{W}_n}^{\mathcal{W}_{n+1}} \underbrace{\mathrm{Ind}_{\mathcal{W}_d \times \mathcal{W}_{n+1-d}}^{\mathcal{W}_{n+1}} U \boxtimes \mathbb{Q}}_{M_{\mathcal{W}}(U)_{n+1}},$$

which define the $\mathrm{FI}_{\mathcal{W}}$–module structure on $M_{\mathcal{W}}(U)$. An alternate, equivalent description of $M_{\mathcal{W}}(U)$ is given in Definition 2.3.

Taking $d = 0$ and U the trivial representation $\mathbb{Q}$, we recover Example 1.5.1. Taking $d = 1$ and $U \cong \mathbb{Q}^1$ recovers Example 1.5.2. For any d, taking U to be the regular representation $k[\mathcal{W}_d]$ recovers Example 1.5.4. The $\mathrm{FI}_{\mathcal{W}}$–modules $M_{\mathcal{W}}(U)$ are discussed in Section 2.3.1.

6. **Example: Zero maps.** Let $\{V_n\}$ be any sequence of non-zero rational $\mathcal{W}_n$–representations, and and let $(I_n)_*$ be the zero maps. These form an $\mathrm{FI}_{\mathcal{W}}$–module that is infinitely generated, with infinite degree of generation.

7. **Example: Torsion and truncated $\mathrm{FI}_{\mathcal{W}}$–modules.** Define $\mathrm{FI}_{\mathcal{W}}$–modules V and U by

$$V_n := \begin{cases} \mathbb{Q}^n, \text{ with } (I_n)_* \text{ the natural inclusions}, \ n < 20 \\ 0, \ n \geq 20 \end{cases}$$

$$U_n := \begin{cases} 0, \ n < 20 \\ \mathbb{Q}^n, \text{ with } (I_n)_* \text{ the natural inclusions}, \ n \geq 20 \end{cases}$$

Then the "torsion" $\mathrm{FI}_{\mathcal{W}}$-module V is finitely generated in degree 1, and the "truncated" $\mathrm{FI}_{\mathcal{W}}$-module U is finitely generated in degree 20.

In contrast to Example 1.5, the sequence of alternating representations and the sequence of regular representations $\mathbb{Q}[\mathcal{W}_n]$, each with their canonical inclusions, do *not* form $\mathrm{FI}_{\mathcal{W}}$–modules; see [Wil14, Examples 3.5 and 3.6].

1.2 Character polynomials in type B/C and D

Let k be a field of characteristic zero. One of our main results is that the sequence of characters of a finitely generated $\mathrm{FI}_{\mathcal{W}}$–module over k is, for n large, equal to a *character polynomial* which does not depend on n. This was proven for symmetric groups in [CEF12, Theorem 2.67], and here we extend these results to the groups D_n and B_n.

Character polynomials for the symmetric groups date back to Murnaghan [Mur51] and Specht [Spe60]; they are described in Macdonald [Mac79, I.7.14]. In Section 4 we introduce character polynomials for the groups B_n and D_n, in two families of *signed* variables. We use the classical results for S_n to derive formulas for the character polynomials of irreducible B_n–representations (Theorem 4.11), and use these formulas to study the characters of $\mathrm{FI}_{\mathcal{W}}$–modules in type B/C and D.

Conjugacy classes of the hyperoctahedral group are classified by *signed cycle type*, see Section 2.1.2 for a description. We define the functions X_r, Y_r on $\coprod_{n=0}^{\infty} B_n$ such that

$$X_r(\omega) \text{ is the number of positive } r\text{–cycles in } \omega,$$
$$Y_r(\omega) \text{ is the number of negative } r\text{–cycles in } \omega.$$

The functions X_r, Y_r are algebraically independent as class functions on $\coprod_{n=0}^{\infty} B_n$, and so they form a polynomial ring $k[X_1, Y_1, X_2, Y_2, \ldots]$ whose elements span the class functions on B_n for each $n \geq 0$.

We prove that the sequence of characters of $\{V_n\}$ associated to any finitely generated FI_{BC}–module or FI_D–module V over a field of characteristic zero are equal to a unique element of $k[X_1, Y_1, X_2, Y_2, \ldots]$ for all n sufficiently large.

Example 1.6. (Signed permutation matrices: A first example of a character polynomial). As an elementary example of a sequence of B_n–representations described by a character polynomial, consider the canonical action of the hyperoctahedral groups B_n on the vector space $\mathbb{Q}^n$ by signed permutation matrices. The trace of a signed permutation matrix σ is

$$\begin{aligned}\mathrm{Tr}(\sigma) &= \#\{1\text{'s on the diagonal of } \sigma\} - \#\{(-1)\text{'s on the diagonal of } \sigma\} \\ &= \#\{\text{ positive one cycles of } \sigma\} - \#\{\text{ negative one cycles of } \sigma\} \\ &= X_1(\sigma) - Y_1(\sigma)\end{aligned}$$

and so the characters χ_n of this sequence are given by the function $\chi_n = X_1 - Y_1$ for all values of n.

The group D_n is canonically realized as the subgroup of this signed permutation matrix group comprising those matrices with an even number of entries equal to

(-1). The character of this representation is the restriction of the character χ_n to the subgroup $D_n \subseteq B_n$, and so again this sequence of characters is equal to the character polynomial $\chi_n = X_1 - Y_1$ for all values of n.

Conjugacy classes of the groups $D_n \subseteq B_n$ are not fully classified by their signed cycle type, due to the existence of certain 'split' classes when n is even; see Section 2.1.3 for details. The functions $\{X_r, Y_r\}$ therefore do not span the space of class functions on any group D_n with n even. We prove, however, that when a sequence of representations $\{V_n\}$ of D_n has the structure of a finitely generated FI_D–module, for n large the characters depend only on the signed cycle type of the classes. Remarkably, the characters associated to $\{V_n\}$ are, for n large, also equal to a character polynomial independent of n.

Theorem 4.16. (Characters of finitely generated $\text{FI}_\mathcal{W}$–modules are eventually polynomial). *Let k be a field of characteristic zero. Suppose that V is a finitely generated FI_{BC}–module with weight $\leq d$ and stability degree $\leq s$, or, alternatively, suppose that V is a finitely generated FI_D–module with weight $\leq d$ such that $\text{Ind}_D^{BC}\, V$ has stability degree $\leq s$. In either case, there is a unique polynomial*

$$F_V \in k[X_1, Y_1, X_2, Y_2, \ldots],$$

independent of n, such that the character of $\mathcal{W}_n$ on V_n is given by F_V for all $n \geq s + d$. The polynomial F_V has degree $\leq d$, with deg(X_i) =deg$(Y_i) = i$.

Weight and stability degree are defined in Sections 2.3.5 and 2.3.6; these quantities are always finite for finitely generated $\text{FI}_\mathcal{W}$–modules and associated induced $\text{FI}_\mathcal{W}$–modules.

Theorem 4.16 generalizes the result of Church–Ellenberg–Farb [CEF12, Theorem 2.67] that the characters of finitely generated FI_A–module are, for n sufficiently large, given by a character polynomial in the class functions X_r on $\coprod_{n=0}^{\infty} S_n$ that takes a permutation σ and returns the number of r–cycles in its cycle type.

In our applications, it remains an open problem to compute the character polynomials in all but a few small degrees. Since we can often establish explicit upper bounds on the degrees and stable ranges of these polynomials, the problem is much more tractable: to find the character polynomials – and so determine the characters for all values of n – it is enough to compute the characters for finitely many specific values of n.

Eventually polynomial dimensions. Suppose that V is a finitely generated $\text{FI}_\mathcal{W}$–module with character polynomial F_V. For each n in the stable range, the dimension $\dim(V_n)$ is given by $F_V(n, 0, 0, 0, \ldots)$, the value of the character polynomial on the identity element in $\mathcal{W}_n$. This has the immediate consequence:

Corollary 4.17. (Polynomial growth of dimension). *Let V be an $FI_{\mathcal{W}}$–module over a field of characteristic zero, and suppose V is finitely generated in degree $\leq d$. Then for large n, dim(V_n) is equal to a polynomial in n of degree at most d. Equality holds for n in the stable range given in Theorem 4.16.*

Although our results on character polynomials in general hold only over fields of characteristic zero, this "eventually polynomial" growth of dimension holds even over positive characteristic, as we show in Theorem 4.20 (stated above). Our proof of Theorem 4.20 uses result in type A proven by Church–Ellenberg–Farb–Nagpal [CEFN14, Theorem 1.2].

1.3 $\text{FI}_{\mathcal{W}}\sharp$–modules

In Section 3 we study *$FI_{BC}\sharp$–modules*, a class of FI_{BC}–modules with additional structure: the FI_{BC} morphisms admit partial inverses. See Definition 3.1 for a complete description. $\text{FI}_{BC}\sharp$–modules mirror the FI$\sharp$–modules ("FI sharp modules") introduced by Church–Ellenberg–Farb [CEF12] in type A.

In Example 1.5, the FI_{BC}–modules in number 1, 2, 3, and 5 all have $\text{FI}_{BC}\sharp$–module structures. Number 6 and 7 cannot be promoted to $\text{FI}_{BC}\sharp$–modules.

The structure of a finitely generated $\text{FI}_{BC}\sharp$–module is highly constrained. We prove in Theorem 3.7 that $\text{FI}_{BC}\sharp$–modules can be decomposed as direct sums of $\text{FI}_{\mathcal{W}}$–modules of the form

$$\left\{ \bigoplus_{m=0} \text{Ind}_{B_m \times B_{n-m}}^{B_n} U_m \boxtimes k \right\}_n = \bigoplus_{m=0} M_{\mathcal{W}}(U_m).$$

As in Example 1.5.5, k denotes the trivial B_{n-m}–representation, and U_m is a B_m–representation, possibly 0. The external tensor product $(U_m \boxtimes k)$ is the k–module $(U_m \otimes_k k)$ as a $(B_m \times B_{n-m})$–representation. This classification result parallels a corresponding statement for FI$\sharp$–modules proven by Church–Ellenberg–Farb [CEF12, Theorem 2.24].

Some consequences of this additional structure: an $\text{FI}_{BC}\sharp$–module finitely generated in degree $\leq d$ has characters equal to a unique character polynomial of degree at most d for *all* values of n, and dimensions given by a polynomial in n of degree at most d for all n. Additional consequences are given in Section 4.5.

1.4 Some applications

$\text{FI}_{\mathcal{W}}$–modules and $\text{FI}_{\mathcal{W}}\sharp$–modules arise naturally throughout geometry and topology, and in Section 5 we use the theory developed here to give results for two such families: the cohomology groups of the pure string motion group $P\Sigma_n$, and the cohomology groups of the hyperplane complements associated to the reflection groups $\mathcal{W}_n$.

Application: the pure string motion group. Let $P\Sigma_n$ be the group of *pure string motions*, motions of n disjoint, unlinked, unknotted, smoothly embedded circles S^1 in $\mathbb{R}^3$. This motion group is a generalization of the pure braid group, and can be realized as the group of *pure symmetric automorphisms* of the free group F_n; see Section 5.1 for a complete definition.

The pure string motion group also appears in the literature under the names the *group of loops*, the *pure untwisted ring group*, the group of *basis-conjugating automorphisms* of the free group, the *Fouxe-Rabinovitch automorphism group* of the free group, and the *Whitehead automorphism group* of the free group. For more background on these groups and their cohomology, see for example Brendle–Hatcher [BH11], Brownstein–Lee [BL93], Dahm [Dah62], Goldsmith [Gol81], Jensen–McCammond–Meier [JMM06], McCool [McC86], or Wilson [Wil12].

Theorem 5.3. *Let k be $\mathbb{Z}$ or $\mathbb{Q}$. The cohomology rings $H^*(P\Sigma_\bullet, k)$ form an $\mathrm{FI}_{BC}\sharp$–module, and a graded FI_{BC}–algebra of finite type, with $H^m(P\Sigma_\bullet, k)$ finitely generated in degree $\leq 2m$. In particular the FI_{BC}–algebra $H^*(P\Sigma_\bullet, \mathbb{Q})$ has slope ≤ 2.*

We recover (with considerably less effort) the main result of our previous paper [Wil12]:

Corollary 5.4 . *For each m, the sequence $\{H^m(P\Sigma_n;\mathbb{Q})\}_n$ of representations of B_n (or S_n) is uniformly representation stable, stabilizing once $n \geq 4m$.*

A consequence of uniform representation stability, which follows from stability for the trivial representation and a transfer argument, is rational homological stability for the string motion group Σ_n. This recovers the rational case of a result of Hatcher and Wahl [HW10, Corollary 1.2]. More details are given in Section 7 of [Wil12].

Another consequence of Theorem 5.3 is the existence of character polynomials. Because these cohomology groups are $\mathrm{FI}_{BC}\sharp$–modules, their characters are equal to the character polynomial for all values of n, and not just n sufficiently large.

Corollary 5.6. *Let k be $\mathbb{Z}$ or $\mathbb{Q}$. Fix an integer $m \geq 0$. The characters of the sequence of B_n–representations $\{H^m(P\Sigma_n;k)\}_n$ are given, for all values of n, by a unique character polynomial of degree $\leq 2m$.*

We compute these character polynomials explicitly in degree 1 and 2:

$$\chi_{H^1(P\Sigma_\bullet;\mathbb{Z})} = X_1^2 - X_1 - Y_1^2 + Y_1$$

$$\begin{aligned}\chi_{H^2(P\Sigma_\bullet;\mathbb{Z})} = {} & 2X_2 + Y_1^2 + 2Y_2^2 - X_1^2Y_1^2 - \frac{3}{2}Y_1^3 + \frac{1}{2}Y_1^4 + X_1^2 - 2X_2^2 - \frac{3}{2}X_1^3 \\ & + \frac{1}{2}X_1^4 + \frac{1}{2}X_1Y_1^2 - X_1Y_2 - X_2Y_1 - Y_1Y_2 + \frac{1}{2}X_1^2Y_1 - X_1X_2 - 2Y_2\end{aligned}$$

It is an open problem to compute these polynomials for larger values of m.

Application: hyperplane complements. Each family of groups $\mathcal{W}_n$ has a canonical action on $\mathbb{R}^n$ by signed permutation matrices; we denote by $\mathscr{A}_{\mathcal{W}}(n)$ the set of complexified hyperplanes fixed by reflections in $\mathcal{W}_n$, and

$$\mathscr{M}_{\mathcal{W}}(n) := \mathbb{C}^n \Big\backslash \bigcup_{H \in \mathscr{A}(n)} H$$

the associated hyperplane complement. See Section 5.2 for explicit descriptions of these spaces, and a brief survey of results on the structure of their cohomology rings. In type A, the space $\mathscr{M}_A(n)$ is precisely the ordered n-point configuration space of $\mathbb{C}$, and Church–Ellenberg–Farb show its cohomology groups are finitely generated $\mathrm{FI}_A\sharp$–modules [CEF12, Theorem 4.7]. Using a presentation for $H^*(\mathscr{M}_{\mathcal{W}}(n);\mathbb{C})$ computed by Brieskorn [Bri73] and Orlik–Solomon [OS80], we generalize the results of [CEF12] to all three families of classical Weyl groups.

Theorem 5.8. *Let $\mathscr{M}_{\mathcal{W}}$ be the complex hyperplane complement associated with the Weyl group $\mathcal{W}_n$ in type A_{n-1}, B_n/C_n, or D_n. In each degree m, the groups $H^m(\mathscr{M}_A(\bullet),\mathbb{C})$ form an $\mathrm{FI}_A\sharp$–module finitely generated in degree $\leq 2m$, and both $H^m(\mathscr{M}_{BC}(\bullet),\mathbb{C})$ and $H^m(\mathscr{M}_D(\bullet),\mathbb{C})$ are $\mathrm{FI}_{BC}\sharp$–modules finitely generated in degree $\leq 2m$.*

Corollary 5.9. *In each degree m, the sequence of cohomology groups $\{H^m(\mathscr{M}_{\mathcal{W}}(n),\mathbb{C})\}_n$ is uniformly representation stable in degree $\leq 4m$.*

Corollary 5.10. *In each degree m, the sequence of characters of the $\mathcal{W}_n$–representations $H^m(\mathscr{M}_{\mathcal{W}}(n),\mathbb{C})$ are given by a unique character polynomial of degree $\leq 2m$ for all n.*

We emphasize that, because these sequences are $\mathrm{FI}_{\mathcal{W}}\sharp$–modules, their characters are equal to the character polynomial for *every* value of n.

Corollary 5.9 recovers the work of Church–Farb [CF13, Theorem 4.1 and 4.6] in types A and B/C. In type A, Theorem 5.8 follows from the work of Church–Ellenberg–Farb [CEF12] on the cohomology of the ordered configuration space of the plane.

Character polynomials and stable decompositions for $H^m(\mathscr{M}_A(\bullet), \mathbb{C})$ are computed in [CEF12] for some small values of m. In Type B/C and D, we can also compute the character polynomials by hand in small degree:

$$\chi_{H^1(\mathscr{M}_D(\bullet),\mathbb{C})} = 2\binom{X_1}{2} + 2\binom{Y_1}{2} + 2X_2 \qquad \chi_{H^1(\mathscr{M}_{BC}(\bullet),\mathbb{C})} = 2\binom{X_1}{2} + 2\binom{Y_1}{2} + 2X_2 + X_1 - Y_1$$

See Section 5.2 for the character polynomials and stable decompositions in degree m is 1 and 2.

1.5 Relationship to the theory of FI–modules

Our theory of $\mathrm{FI}_\mathcal{W}$–modules has very close parallels to work of Church, Ellenberg, Farb, and Nagpal [CEF12, CEFN14] on the symmetric groups, which we aim to highlight throughout this paper. Working with the other classical Weyl groups, however, we do encounter some new obstacles and some new phenomena. We enumerate some differences here:

Character polynomials in type B/C. The existence of character polynomials for finitely generated FI_A–modules follows immediately from representation stability and classical results in algebraic combinatorics: the formula for the character polynomial of the irreducible S_n–representation $V(\lambda)_n$ appear in texts such as MacDonald [Mac79]. The achievement of [CEF12] here was uncovering this (regrettably little-known) formula and recognizing its implications for the study of FI_A–modules. The analogous formulas for the irreducible B_n–representations are less readily available, however, and we compute these in Section 4.2. These signed character polynomials now involve two sets of variables X_r and Y_r, corresponding to the positive and negative cycles for these groups.

Character polynomials in type D. Given the classification of conjugacy classes in type D (Section 2.1.3), and the existence of 'split' classes that could not be characterized by signed cycle type, we had not expected an analogue of character polynomials to exist for sequences of D_n–representations, except in exceptional cases. A finitely generated FI_D–module *does* have characters equal, for large n, to a character polynomial. We establish this existence result by realizing the tail of a finitely generated FI_D–module V as the restriction of an FI_{BC}–module, using properties of categorical induction Ind_D^{BC}.

A category $\mathrm{FI}_D\sharp$? There does not appear to be a suitable analogue of FI$\sharp$ for the category FI_D; see Remark 3.2. Fortunately, and perhaps not by coincidence, the applications in type D where we have expected this extra structure, such as the cohomology groups of the hyperplane complements $\mathscr{M}_D(n)$, turned out to be restrictions of $\text{FI}_{BC}\sharp$–modules to $\text{FI}_D \subseteq \text{FI}_{BC}$.

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2 Background

2.1 The Weyl groups of classical type and their representation theory

We briefly summarize the rational representation theory of the three families of Weyl groups, and the associated notation used in this paper. A more detailed review, with additional references, is given in [Wil14, Section 2].

2.1.1 The symmetric group S_n

Given a partition $\lambda \vdash n$, $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_r)$, we denote the *parts* of the partition by λ_i and index them in decreasing order $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_r$. We write $|\lambda| = n$ to indicate the size of the partition, and write $\ell(\lambda)$ to denote the *length* of λ, the number of parts.

The rational irreducible representations of the symmetric group S_n are classified by partitions of n (see for example Fulton–Harris [FH04]) and we write V_λ to denote the S_n–representation associated to the partition λ.

Given a partition $\lambda \vdash m$ and an integer $n \geq \lambda_1 + m$, define the *padded partition*

$$\lambda[n] := ((n-m), \lambda_1, \lambda_2, \ldots, \lambda_t).$$

We denote by $V(\lambda)_n$ the irreducible S_n–representation associated to $\lambda[n]$, that is,

$$V(\lambda)_n := \begin{cases} V_{\lambda[n]} & (n-m) \geq \lambda_1, \\ 0 & \text{otherwise.} \end{cases}$$

2.1.2 The hyperoctahedral group B_n

Recall that the hyperoctahedral group (or signed permutation group) B_n is the group of generalized permutation matrices with nonzero entries ± 1; equivalently, $B_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ is the symmetry group of the set $\{\{-1,1\},\{-2,2\},\ldots,\{-n,n\}\}$. We frequently consider B_n as a subgroup of the symmetric group on the set

$$\Omega = \{-1, 1, -2, 2, \ldots, -n, n\},$$

and write signed permutations in the corresponding cycle notation.

The rational representation theory of B_n. Recall that the irreducible rational B_n–representations are classified by *double partitions* of n, ordered pairs of partitions $\lambda = (\lambda^+, \lambda^-)$ with $|\lambda^+| + |\lambda^-| = n$, as follows.

Let $V_{(\lambda^+,\varnothing)}$ denote the B_n–representation pulled back from S_n–representation V_{λ^+}, and denote

$$V_{(\varnothing,\lambda^-)} := V_{(\lambda^-,\varnothing)} \otimes \mathbb{Q}^{\varepsilon},$$

where $\mathbb{Q}^{\varepsilon}$ is the one-dimensional representation given by the character $\varepsilon : B_n \to B_n/D_n = \{\pm 1\}$. Then for $\lambda^+ \vdash m$ and $\lambda^- \vdash (n-m)$ we define

$$V_{(\lambda^+,\lambda^-)} := \mathrm{Ind}_{B_m \times B_{n-m}}^{B_n} V_{(\lambda^+,\varnothing)} \boxtimes V_{(\varnothing,\lambda^-)},$$

where $\boxtimes$ again denotes the external tensor product of the B_m–representation $V_{(\lambda^+,\varnothing)}$ with the B_{n-m}–representation $V_{(\varnothing,\lambda^-)}$. The rational irreducible B_n–representations are precisely the set

$$\left\{ V_{(\lambda^+,\lambda^-)} \;\middle|\; (\lambda^+,\lambda^-) \text{ is a double partition of } n \right\}.$$

For a double partition $\lambda = (\lambda^+, \lambda^-)$ with $\lambda^+ \vdash \ell$ and $\lambda^- \vdash m$, we define the *padded double partition* $\lambda[n] := (\lambda^+[n-m], \lambda^-)$ associated to $\lambda = (\lambda^+, \lambda^-)$. We write $V(\lambda)_n$ or $V(\lambda^+,\lambda^-)_n$ to denote the irreducible B_n–representation

$$V(\lambda)_n := \begin{cases} V_{\lambda[n]} & (n-m) \geq \lambda_1^+, \\ 0 & \text{otherwise.} \end{cases}$$

The conjugacy classes of B_n. The conjugacy classes of B_n are classified by *signed cycle type*. Each element of B_n decomposes into a product of *cycles*; a factor is called an *r–cycle* if it maps to an r–cycle in S_n under the natural surjection. An r–cycle is *positive* if its r^{th} power is the identity, equivalently, if it reverses the sign of an even number of digits $\{\pm 1, \ldots, \pm n\}$. An r–cycle is *negative* if its r^{th} power is the product of r involutions $(-i\ i)$, equivalently, if it reverses the sign of an odd number of digits. For example, $(1)(-1)$ is a positive 1–cycle, and $(-1\ 1)$ is a negative 1–cycle.

We designate the cycle type of a signed permutation in B_n by a double partition (ν^+, ν^-) of n, where the parts of ν^+ are the lengths of the positive cycles and the parts of ν^- are the lengths of the negative cycles.

2.1.3 The even-signed permutation group D_n

The even-signed permutation group D_n is the subgroup of signed permutation matrices B_n of matrices that have an even number of entries equal to -1.

The rational representation theory of D_n.

Given an irreducible representation $V_{(\lambda^+,\lambda^-)}$ of B_n with $\lambda^+ \neq \lambda^-$, its restriction to D_n is also irreducible. We denote this irreducible D_n–representation by

$$V_{\{\lambda^+,\lambda^-\}} := \mathrm{Res}_{D_n}^{B_n} V_{(\lambda^+,\lambda^-)} \cong \mathrm{Res}_{D_n}^{B_n} V_{(\lambda^-,\lambda^+)}, \qquad \lambda^+ \neq \lambda^-.$$

These D_n–representations are nonisomorphic for each distinct set of partitions $\{\lambda^+, \lambda^-\}$. When n is even, for any partition $\lambda \vdash \frac{n}{2}$, the irreducible B_n–representation $V_{(\lambda,\lambda)}$ restricts to a sum of two nonisomorphic irreducible D_n–representations of equal dimension. The irreducible rational representations of D_n are therefore classified by the set

$$\left\{\{\lambda^+, \lambda^-\} \;\mid\; \lambda^+ \neq \lambda^-,\ |\lambda^+| + |\lambda^-| = n\right\} \coprod \left\{(\lambda, \pm) \;\middle|\; |\lambda| = \frac{n}{2}\right\},$$

The 'split' irreducible representations $V_{(\lambda,+)}$ and $V_{(\lambda,-)}$ only arise when n is even.

Given an (ordered) double partition $\lambda = (\lambda^+, \lambda^-)$ with $\lambda^+ \vdash \ell$ and $\lambda^- \vdash m$, we write $V(\lambda)_n$ to denote the D_n–representation $V(\lambda)_n := \mathrm{Res}_{D_n}^{B_n} V(\lambda)_n$. Explicitly, $V(\lambda)_n$ is the D_n–representation

$$V(\lambda)_n = \begin{cases} V_{\{\lambda^+[n-m],\, \lambda^-\}} & (n-m) \geq \lambda_1^+ \text{ and } \lambda^+[n-m] \neq \lambda^-, \\ V_{\{\lambda^-,\, +\}} \oplus V_{\{\lambda^-,\, -\}} & (n-m) \geq \lambda_1^+ \text{ and } \lambda^+[n-m] = \lambda^-, \\ 0 & \text{otherwise.} \end{cases}$$

The D_n–representation $V(\lambda)_n$ is irreducible for all but at most one value of n.

We remark that, in contrast to the sequence of S_n or B_n representations $V(\lambda)_n$, knowing the D_n–representation $V(\lambda)_n$ for a single value of n may not be enough to determine $V(\lambda)_{n+1}$, as we cannot distinguish the partitions $\lambda^+[n-m]$ and λ^-.

The conjugacy classes of D_n. As with B_n, each element of D_n decomposes into a product of signed cycles; by definition each element must have an even number of negative cycles. Signed cycle type is a D_n conjugacy class invariant, and it nearly classifies the conjugacy classes, with one qualification: when n is even, the elements for which all cycles are positive and have even length are now split between two conjugacy classes.

2.2 Representation stability

Representation stability was introduced by Church–Farb [CF13] for a variety of families of groups G_n including S_n and B_n. In [Wil14, Section 2.2] we additionally defined representation stability for sequences of D_n–representations. We recall these definitions here.

Definition 2.1. (Representation stability). Let $\mathcal{W}_n$ be one of the families of classical Weyl groups, and suppose $\{V_n\}$ is a sequence of finite-dimensional $\mathcal{W}_n$–representations over characteristic zero with maps $\phi_n : V_n \to V_{n+1}$. The sequence $\{V_n, \phi_n\}$ is *consistent* if ϕ_n is equivariant with respect to the action of $\mathcal{W}_n$ on V_n and of $\mathcal{W}_n \subseteq \mathcal{W}_{n+1}$ on V_{n+1}.

A consistent sequence $\{V_n, \phi_n\}$ is *representation stable* if it satisfies three properties:

I. ***Injectivity.*** The maps $\phi_n : V_n \to V_{n+1}$ are injective for $n >> 0$.

II. ***Surjectivity.*** The image $\phi_n(V_n)$ generates V_{n+1} as a $k[\mathcal{W}_{n+1}]$–module for $n >> 0$.

III. ***Multiplicities.*** Decompose V_n into irreducible $\mathcal{W}_n$–representations:

$$V_n = \bigoplus_{\lambda} c_{\lambda,n} V(\lambda)_n.$$

For each λ there exists some N_λ such that the multiplicity $c_{\lambda,n}$ of $V(\lambda)_n$ is constant for $n \geq N_\lambda$.

The sequence is *uniformly* representation stable if N_λ can be chosen independently of λ.

A main result of [Wil14] is that for rational $\mathrm{FI}_\mathcal{W}$–modules, finite generation is equivalent to uniform representation stability.

Theorem 2.2. [Wil14, Theorem 4.22] *Let k be a field of characteristic zero. An $\mathrm{FI}_\mathcal{W}$–module V is finitely generated if and only if $\{V_n\}$ is uniformly representation stable with respect to the maps induced by the natural inclusions $I_n : \mathbf{n} \hookrightarrow (\mathbf{n+1})$.*

Details (including bounds on the stable range) are given in Section 2.3.7.

2.3 Summary of terminology and foundations for $\mathrm{FI}_\mathcal{W}$–modules

In this section we summarize the main definitions and foundational results developed in [Wil14] on $\mathrm{FI}_\mathcal{W}$–module theory.

2.3.1 The $\mathrm{FI}_\mathcal{W}$–modules $M_\mathcal{W}(\mathbf{m})$ and $M_\mathcal{W}(U)$

Definition 2.3. (The $\mathrm{FI}_\mathcal{W}$–modules $M_\mathcal{W}(\mathbf{m})$ and $M_\mathcal{W}(U)$). Let $\mathcal{W}_n$ denote S_n, B_n, or D_n. For fixed integer $m \geq 0$, define $M_\mathcal{W}(\mathbf{m})$ to be the $\mathrm{FI}_\mathcal{W}$–module over k such that

$$M_\mathcal{W}(\mathbf{m})_n := k\big[\mathrm{Hom}_{\mathrm{FI}_\mathcal{W}}(\mathbf{m}, \mathbf{n})\big]$$

with an action of $\mathcal{W}_n$ by postcomposition.

We can identify $M_{\mathcal{W}}(\mathbf{m})_n$ with the k–span of the set

$$\left\{ \Big(f(1), f(2), \ldots, f(m) \Big) \subseteq \mathbf{n} \;\mid\; f : \mathbf{m} \to \mathbf{n} \text{ an } \mathrm{FI}_{\mathcal{W}} \text{ morphism} \right\}.$$

For each n we have an isomorphism of $\mathcal{W}_n$–represntations

$$M_{\mathcal{W}}(\mathbf{m})_n \cong \mathrm{Ind}_{\mathcal{W}_{n-m}}^{\mathcal{W}_n} k;$$

here k is the trivial $\mathcal{W}_m$–representation.

Given a $\mathcal{W}_m$–representation U, we define $M_{\mathcal{W}}(U)$ to be the $\mathrm{FI}_{\mathcal{W}}$–module

$$M_{\mathcal{W}}(U)_n := M_{\mathcal{W}}(\mathbf{m})_n \otimes_{k[\mathcal{W}_m]} U.$$

In particular $M_{\mathcal{W}}(\mathbf{m}) \cong M_{\mathcal{W}}(k[\mathcal{W}_m])$.

Let $\mathrm{FB}_{\mathcal{W}}$ denote the wide subcategory of $\mathrm{FI}_{\mathcal{W}}$ consisting of all objects and all endomorphisms. Denote by $\mathrm{FB}_{\mathcal{W}}$–Mod the category of functors

$$\mathrm{FB}_{\mathcal{W}} \longrightarrow k\text{–Mod};$$

the objects are sequences of $\mathcal{W}_m$–representations (with no additional maps) and the morphisms are sequences of $\mathcal{W}_m$–equivariant maps.

We extend $M_{\mathcal{W}}$ to a functor on the category $\mathrm{FB}_{\mathcal{W}}$–Mod

$$\begin{aligned} M_{\mathcal{W}} : \mathrm{FB}_{\mathcal{W}}\text{–Mod} &\longrightarrow \mathrm{FI}_{\mathcal{W}}\text{–Mod} \\ U_m &\longmapsto \mu_m(U_m). \end{aligned}$$

Given an $\mathrm{FI}_{\mathcal{W}}$–module V and any subset $S = \{v_i\} \subseteq \coprod_{n \geq 0} V_n$, with $v_i \in V_{m_i}$, there is a unique map of $\mathrm{FI}_{\mathcal{W}}$–module

$$\bigoplus_{i=1} M_{\mathcal{W}}(\mathbf{m}_i) \longrightarrow V$$

$$f \longmapsto f_*(v_i) \qquad f \in \mathrm{Hom}_{\mathcal{W}}(\mathbf{m}_i, \mathbf{n}), \text{ the basis for } M_{\mathcal{W}}(\mathbf{m}_i)_n$$

from $\bigoplus_{i=1} M_{\mathcal{W}}(\mathbf{m}_i)$ onto the $\mathrm{FI}_{\mathcal{W}}$–submodule *generated by S*, the smallest submodule of V that contains S. An $\mathrm{FI}_{\mathcal{W}}$–module V is finitely generated if and only if it is the quotient of a finite sum of $\mathrm{FI}_{\mathcal{W}}$–modules $\bigoplus_{i=1}^{\ell} M_{\mathcal{W}}(\mathbf{m}_i)$.

Definition 2.4. (Finite Presentation; Relation degree). Let V be a finitely generated $\mathrm{FI}_{\mathcal{W}}$–module. Then V is *finitely presented* with *relation degree r* if there is a surjection

$$\bigoplus_{i=1}^{\ell} M_{\mathcal{W}}(\mathbf{m}_i) \twoheadrightarrow V$$

with a kernel finitely generated in degree at most r.

The Noetherian property [Wil14, Theorem 4.21] implies that all finitely generated $\mathrm{FI}_{\mathcal{W}}$–modules are finitely presented.

2.3.2 The Noetherian property

A critical property of $\mathrm{FI}_{\mathcal{W}}$–modules that underlies all our major results is that the category of $\mathrm{FI}_{\mathcal{W}}$–modules over a Noetherian ring is Noetherian. Church–Ellenberg–Farb–Nagpal prove this result for FI_A–modules [CEFN14, Theorem 1.1], and we use their work to prove it more generally:

Theorem 2.5. [Wil14, Theorem 4.21] **($\mathrm{FI}_{\mathcal{W}}$–modules are Noetherian).** *Let k be a Noetherian ring. Any sub–$\mathrm{FI}_{\mathcal{W}}$–module of a finitely generated $\mathrm{FI}_{\mathcal{W}}$–module over k is itself finitely generated.*

2.3.3 The functor H_0

Definition 2.6. (The functor H_0). As in [CEF12, Definition 2.18], we define the functor H_0 by

$$H_0 : \mathrm{FI}_{\mathcal{W}}\text{–Mod} \longrightarrow \mathrm{FB}_{\mathcal{W}}\text{–Mod}$$
$$(H_0(V))_n = V_n \Big/ \left(\mathrm{span}_V \Big(\coprod_{k<n} V_k \Big) \right)_n$$

$H_0(V)$ is a minimal set of $\mathcal{W}_m$–representations to generate the $\mathrm{FI}_{\mathcal{W}}$–module V. We can put an $\mathrm{FI}_{\mathcal{W}}$–module structure on the $\mathcal{W}_n$–representations $(H_0(V))_n$ by letting I_n act by 0 for all n; it is the largest quotient of V where all $\mathrm{FI}_{\mathcal{W}}$ morphisms f between distinct objects act by zero. We denote this $\mathrm{FI}_{\mathcal{W}}$–module by $H_0(V)^{\mathrm{FI}_{\mathcal{W}}}$.

H_0 is a left inverse to $M_{\mathcal{W}}$, that is, given $U = \{U_m\}_m$ we have $H_0(M_{\mathcal{W}}(U))_m = U_m$ for all m. We will see in Section 3 that additionally $M_{\mathcal{W}}(H_0(V)) = V$ when V has the additional structure of an *$\mathrm{FI}_{\mathcal{W}}\sharp$–module*.

There are surjections

$$M_{\mathcal{W}}(H_0(V)) \twoheadrightarrow V \qquad \text{and} \qquad V \twoheadrightarrow H_0(V)^{\mathrm{FI}_{\mathcal{W}}}.$$

2.3.4 Restriction and induction of $\mathrm{FI}_{\mathcal{W}}$–modules

The inclusions of categories $\mathrm{FI}_A \subseteq \mathrm{FI}_D \subseteq \mathrm{FI}_{BC}$ enable us to define restriction and induction operations on the corresponding categories of $\mathrm{FI}_{\mathcal{W}}$–modules. Both restriction and induction preserve finite generation of $\mathrm{FI}_{\mathcal{W}}$–modules, a fact that we use to recover results in type B/C and D from work of Church–Ellenberg–Farb–Nagpal

[CEFN14] in type A. We use additional properties of these operations to develop theory for FI_D–modules using results in type B/C.

Definition 2.7. (Restriction). Consider Weyl group families $\mathcal{W}_n$ and $\overline{\mathcal{W}}_n$ with $\mathcal{W}_n \subseteq \overline{\mathcal{W}}_n$. Then there is an inclusion $\mathrm{FI}_{\mathcal{W}} \hookrightarrow \mathrm{FI}_{\overline{\mathcal{W}}}$, and given any $\mathrm{FI}_{\overline{\mathcal{W}}}$–module V we denote by $\mathrm{Res}_{\mathcal{W}}^{\overline{\mathcal{W}}} V$ the $\mathrm{FI}_{\mathcal{W}}$–module obtained by restricting the functor V to the subcategory $\mathrm{FI}_{\mathcal{W}}$.

Proposition 2.8. [Wil14, Proposition 3.22] **(Restriction preserves finite generation).** *For each family of Weyl groups $\mathcal{W}_n \subseteq \overline{\mathcal{W}}_n$, the restriction $\mathrm{Res}_{\mathcal{W}}^{\overline{\mathcal{W}}} V$ of a finitely generated $\mathrm{FI}_{\overline{\mathcal{W}}}$–module V is finitely generated as an $\mathrm{FI}_{\mathcal{W}}$–module. Specifically,*

1. *Given an FI_{BC}–module V finitely generated in degree $\leq m$, $\mathrm{Res}_A^{BC} V$ is finitely generated as an FI_A–module in degree $\leq m$.*
2. *Given an FI_{BC}–module V finitely generated in degree $\leq m$, $\mathrm{Res}_D^{BC} V$ is finitely generated as an FI_D–module in degree $\leq m$.*
3. *Given an FI_D–module V finitely generated in degree $\leq m$, $\mathrm{Res}_A^{D} V$ is finitely generated as an FI_A–module in degree $\leq (m+1)$.*

For categories $\mathrm{FI}_{\mathcal{W}} \hookrightarrow \mathrm{FI}_{\overline{\mathcal{W}}}$, the restriction functor

$$\mathrm{Res}_{\mathcal{W}}^{\overline{\mathcal{W}}} : \mathrm{FI}_{\overline{\mathcal{W}}}\text{–Mod} \longrightarrow \mathrm{FI}_{\mathcal{W}}\text{–Mod}$$

has a left adjoint, the induction functor

$$\mathrm{Ind}_{\mathcal{W}}^{\overline{\mathcal{W}}} : \mathrm{FI}_{\mathcal{W}}\text{–Mod} \longrightarrow \mathrm{FI}_{\overline{\mathcal{W}}}\text{–Mod}.$$

We recall that the adjunction comes with a *unit* map, a natural transformation

$$\eta : id \longrightarrow (\mathrm{Res}_{\mathcal{W}}^{\overline{\mathcal{W}}} \mathrm{Ind}_{\mathcal{W}}^{\overline{\mathcal{W}}})$$
$$\eta_V : V \longrightarrow \mathrm{Res}_{\mathcal{W}}^{\overline{\mathcal{W}}}(\mathrm{Ind}_{\mathcal{W}}^{\overline{\mathcal{W}}} V)$$

and this map defines a natural bijection

$$\left\{ \begin{array}{c} \mathrm{FI}_{\mathcal{W}} \text{ –Module Maps} \\ V \longrightarrow \mathrm{Res}_{\mathcal{W}}^{\overline{\mathcal{W}}} U \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \mathrm{FI}_{\overline{\mathcal{W}}}\text{–Module Maps} \\ \mathrm{Ind}_{\mathcal{W}}^{\overline{\mathcal{W}}} V \longrightarrow U \end{array} \right\}$$

We can describe the induced functors $\mathrm{Ind}_{\mathcal{W}}^{\overline{\mathcal{W}}} V$ explicitly using the theory of Kan extensions; see Mac Lane [ML98, Chapter 10.4].

Definition 2.9. (Induction). Given an $\mathrm{FI}_{\mathcal{W}}$–module V, and an inclusion of categories $\mathrm{FI}_{\mathcal{W}} \hookrightarrow \mathrm{FI}_{\overline{\mathcal{W}}}$, we define the *induced* $FI_{\overline{\mathcal{W}}}$*–module* $\mathrm{Ind}_{\mathcal{W}}^{\overline{\mathcal{W}}} V$ by

$$(\mathrm{Ind}_{\mathcal{W}}^{\overline{\mathcal{W}}} V)_n = \bigoplus_{r \leq n} M_{\overline{\mathcal{W}}}(\mathbf{r})_n \otimes_k V_r \Big/ \langle \quad f \otimes g_*(v) = (f \circ g) \otimes v \quad | \quad g \text{ is an } \mathrm{FI}_{\mathcal{W}} \text{ morphism} \rangle.$$

with the action of $h \in \mathrm{Hom}_{\overline{\mathcal{W}}}(\mathbf{m}, \mathbf{n})$ by $h_* : g \otimes v \longmapsto (h \circ g) \otimes v$.

Given categories $\mathrm{FI}_{\mathcal{W}} \subseteq \mathrm{FI}_{\overline{\mathcal{W}}}$ and fixed $m \geq 0$, there is an isomorphism of $\mathrm{FI}_{\overline{\mathcal{W}}}$–modules

$$\mathrm{Ind}_{\mathcal{W}}^{\overline{\mathcal{W}}} M_{\mathcal{W}}(\mathbf{m}) \cong M_{\overline{\mathcal{W}}}(\mathbf{m}).$$

A critical property of induction from FI_D to FI_{BC} was proven in [Wil14, Theorem 3.30].

Theorem 2.10. [Wil14, Theorem 3.30] **($V_n \cong (\mathbf{Res}_D^{BC} \mathbf{Ind}_D^{BC} V)_n$ for n large).** *Suppose V is an FI_D–module finitely generated in degree $\leq m$. Then*

$$V_n \xrightarrow{\cong} (Res_D^{BC}\ Ind_D^{BC} V)_n$$

is an isomorphism of D_n–representations for all $n > m$. In particular, every finitely generated FI_D–module V is, for n greater than its degree of generation, the restriction of an FI_{BC}–module.

2.3.5 The weight of an $\mathrm{FI}_{\mathcal{W}}$–module

Definition 2.11. (Weight). Let k be a field of characteristic zero. An FI_A–module V over k has *weight* $\leq d$ if every irreducible constituent $V(\lambda)_n$ of V_n has $|\lambda| \leq d$ for all $n \geq 0$. Similarly an FI_{BC}–module V over k has *weight* $\leq d$ if for all n every irreducible constituent $V(\lambda)_n = V(\lambda^+, \lambda^-)_n$ of V_n satisfies $|\lambda^+| + |\lambda^-| \leq d$. Finally we say an FI_D–module V has *weight* $\leq d$ if the FI_{BC}–module $\mathrm{Ind}_D^{BC} V$ does.

An $\mathrm{FI}_{\mathcal{W}}$–module V has *finite weight* if it is of weight $\leq d$ for some integer $d \geq 0$. The minimum such d is the *weight* of V, denoted weight(V). We say that the *weight of a Young diagram* $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_\ell)$ is

$$\lambda_1 + \cdots + \lambda_\ell = |\lambda| - \lambda_0.$$

For example, the $\mathrm{FI}_{\mathcal{W}}$–module $M_{\mathcal{W}}(\mathbf{m})$ has weight m. Since the weight of an $\mathrm{FI}_{\mathcal{W}}$–module V is an upper bound on the weight of all of its quotients, this implies that the weight of an $\mathrm{FI}_{\mathcal{W}}$–module is bounded by its degree of generation.

Theorem 2.12. *[Wil14, Theorem 4.4]* **(Degree of generation bounds weight).** *Suppose that V is an $FI_{\mathcal{W}}$–module over a field of characteristic zero. If V is finitely generated in degree $\leq m$, then weight(V) $\leq m$.*

2.3.6 Shifted $\mathrm{FI}_{\mathcal{W}}$–modules, coinvariants, and stability degree

We define the shift and coinvariant functors on $\mathrm{FI}_{\mathcal{W}}$–modules, and use these operations to define the stability degree of an $\mathrm{FI}_{\mathcal{W}}$–module. The stability degree featured in the proof of representation stability for finitely generated $\mathrm{FI}_{\mathcal{W}}$–modules, and in Section 4 it will be used to bound the stable range of the associated character polynomials.

Definition 2.13. (Shifts $\amalg_{[-a]}$; Shifted $\mathrm{FI}_{\mathcal{W}}$–modules) For each $a \geq 0$, define the functor

$$\begin{aligned} \amalg_{[-a]} : \mathrm{FI}_{\mathcal{W}} &\longrightarrow \mathrm{FI}_{\mathcal{W}} \\ \mathbf{n} &\longmapsto (\mathbf{n}+\mathbf{a}) \\ \{f : \mathbf{m} \to \mathbf{n}\} &\longmapsto \{\amalg_{[-a]}(f) : (\mathbf{m}+\mathbf{a}) \to (\mathbf{n}+\mathbf{a})\} \end{aligned}$$

where

$$\amalg_{[-a]}(f) \text{ maps } \begin{cases} d \mapsto d & \text{if } d \leq a, \\ (d+a) \mapsto (f(d)+a). \end{cases}$$

Then for an $\mathrm{FI}_{\mathcal{W}}$–module $V : \mathrm{FI}_{\mathcal{W}} \to k$–Mod, we define the *shifted* $\mathrm{FI}_{\mathcal{W}}$–module $S_{+a}V$ by

$$S_{+a}V = V \circ \amalg_{[-a]}.$$

As a $\mathcal{W}_n$–representation, $(S_{+a}V)_n$ is the restriction of V_{n+a} from $\mathcal{W}_{n+a}$ to the copy of $\mathcal{W}_n$ acting on $\{\pm(1+a), \ldots \pm(n+a)\} \subseteq (\mathbf{n}+\mathbf{a})$.

Definition 2.14. (Coinvariants functor τ; The graded $k[T]$–module $\Phi_a(V)$). We define the functor τ, as follows:

$$\begin{aligned} \tau : \mathrm{FI}_{\mathcal{W}}\text{–Mod} &\longrightarrow \mathrm{FI}_{\mathcal{W}}\text{–Mod} \\ V &\longmapsto (\tau V)_n = (V_n)_{\mathcal{W}_n} := k \otimes_{k[\mathcal{W}_n]} V_n \end{aligned}$$

The spaces $(\tau V)_n$ are the $\mathcal{W}_n$–coinvariants of V_n, the largest quotient of V_n on which $\mathcal{W}_n$ acts trivially. Over characteristic zero it is isomorphic to its invariant subspace $(V_n)^{\mathcal{W}_n}$.

Then, for fixed integer $a \geq 0$, we define

$$\begin{aligned} \Phi_a : \mathrm{FI}_{\mathcal{W}}\text{–Mod} &\longrightarrow k[T]\text{–Mod} \\ V &\longmapsto \bigoplus_{n\geq 0} (\tau \circ S_{+a}V)_n = \bigoplus_{n \geq 0} (V_{n+a})_{\mathcal{W}_n} \end{aligned}$$

$\Phi_a(V)$ is a $k[T]$–module under the action of T by the maps $(V_{n+a})_{\mathcal{W}_n} \to (V_{n+1+a})_{\mathcal{W}_{n+1}}$ induced by the maps $(I_{n+a})_* : V_{n+a} \to V_{n+1+a}$.

Definition 2.15. (Injectivity degree; Surjectivity degree; Stability degree). An $\mathrm{FI}_{\mathcal{W}}$–module V has *injectivity degree* $\leq s$ (respectively, *surjectivity degree* $\leq s$) if for all $a \geq 0$ and $n \geq s$, the map $\Phi_a(V)_n \to \Phi_a(V)_{n+1}$ induced by T is injective (respectively, surjective). We call the minimum such s the *injectivity degree* (respectively, the *surjectivity degree*).

We say that V has *stability degree* $\leq s$ if for every $a \geq 0$ and $n \geq s$, the map $\Phi_a(V)_n \to \Phi_a(V)_{n+1}$ induced by T is an isomorphism of k–modules, equivalently, an isomorphism of $\mathcal{W}_a$–representations. Explicitly, V has stability degree $\leq s$ if

$$(V_{n+a})_{\mathcal{W}_n} \cong (V_{n+1+a})_{\mathcal{W}_{n+1}} \qquad \text{for every } a \geq 0 \text{ and all } n \geq s.$$

The *stability degree* is the minimum such s.

2.3.7 Representation stability of finitely generated $\mathrm{FI}_{\mathcal{W}}$–modules

Using the concept of stability degree, we prove in [Wil14, Section 4.4] that for $\mathrm{FI}_{\mathcal{W}}$–modules over characteristic zero, finite generation is equivalent to uniform representation stability. Our first result:

Theorem 2.16. [Wil14, Theorem 4.26] **(Finitely generated $\mathrm{FI}_{\mathcal{W}}$–modules are uniformly representation stable).** *Suppose that k is a characteristic zero field, and that V is an FI_{BC}–module with weight $\leq d$ and stability degree N. Then $\{V_n\}$ is uniformly representation stable with respect to the maps $\phi_n : V_n \to V_{n+1}$ induced by the natural inclusions $I_n : \mathbf{n} \hookrightarrow (\mathbf{n+1})$. The sequences stabilizes for $n \geq N + d$.*

Our analysis of stability degree allows us to recasts the bounds in terms of the generation and relation degree. Using properties of induction and restriction of $\mathrm{FI}_{\mathcal{W}}$–modules, we can extend these representation stability results to FI_D–modules.

Theorem 2.17. [Wil14, Theorem 4.27] **(Finitely generated $\mathrm{FI}_{\mathcal{W}}$–modules are uniformly representation stable).** *Suppose that k is a field of characteristic zero, and $\mathcal{W}_n$ is S_n, D_n or B_n. Let V be a finitely generated $FI_{\mathcal{W}}$–module of weight $\leq d$, generation degree $\leq g$ and relation degree $\leq r$; when $\mathcal{W}_n$ is D_n, take r instead to be an upper bound on the relation degree of $\mathrm{Ind}_D^{BC}\, V$. Then, $\{V_n\}$ is uniformly representation stable with respect to the maps induced by the natural inclusions $I_n : \mathbf{n} \to (\mathbf{n+1})$, stabilizing once $n \geq \max(g, r) + d$; when $\mathcal{W}_n$ is D_n and $d = 0$ we need the additional condition that $n \geq g + 1$.*

The converse statement follows easily from "surjectivity" criterion for representation stability.

Theorem 2.18. [Wil14, Theorem 4.28] **(Uniformly representation stable $\mathrm{FI}_{\mathcal{W}}$–modules are finitely generated).** *Suppose conversely that V is an $FI_{\mathcal{W}}$–module, and that $\{V_n, (I_n)_*\}$ is uniformly representation stable for $n \geq N$. Then V is finitely generated in degree $\leq N$.*

2.3.8 Tensor products of $\mathrm{FI}_{\mathcal{W}}$–modules and graded $\mathrm{FI}_{\mathcal{W}}$–algebras

Definition 2.19. (Tensor product of $\mathrm{FI}_{\mathcal{W}}$–modules). Given $\mathrm{FI}_{\mathcal{W}}$–modules V and W, the *tensor product* $V \otimes W$ is the $\mathrm{FI}_{\mathcal{W}}$–module by $(V \otimes W)_n = V_n \otimes W_n$ with the diagonal action of the $\mathrm{FI}_{\mathcal{W}}$–morphisms.

Proposition 2.20. [Wil14, Proposition 5.2]**(Tensor products respect finite generation).** *If V and W are finitely generated $\mathrm{FI}_{\mathcal{W}}$–modules, then $V \otimes W$ is finitely generated. If V is generated in degree $\leq m$ and W in degree $\leq m'$, then $V \otimes W$ is generated in degree $\leq m + m'$. Over a field of characteristic zero, the weight of $(V \otimes W)$ is at most weight(V) + weight(W).*

Definition 2.21. (Graded $\mathrm{FI}_{\mathcal{W}}$–modules; Finite type; Slope). A *graded $\mathrm{FI}_{\mathcal{W}}$–module* $V = \oplus_i V^i$ is a functor from $\mathrm{FI}_{\mathcal{W}}$ to the category of graded k–modules; each graded piece V^i is an $\mathrm{FI}_{\mathcal{W}}$–module. We say V has *finite type* if V^i is a finitely generated for all i.

Suppose k is a field of characteristic zero. Let V be a graded $\mathrm{FI}_{\mathcal{W}}$–module over k supported in nonnegative degrees. We define the *slope* of V to be $\leq m$ if for all i the $\mathrm{FI}_{\mathcal{W}}$–module V^i has weight $\leq m \cdot i$.

The tensor product of graded $\mathrm{FI}_{\mathcal{W}}$–modules $U = \oplus_i U^i$ and $W = \oplus_j W^j$ is the graded $\mathrm{FI}_{\mathcal{W}}$–module

$$U \otimes W = \bigoplus_{\ell} (U \otimes W)^{\ell} := \bigoplus_{\ell} \left(\bigoplus_{i+j=\ell} (U^i \otimes W^j) \right).$$

Suppose U and W are supported in nonnegative degrees with $U_0 \cong W_0 \cong M_{\mathcal{W}}(\mathbf{0})$. If U and W have finite type, then $U \otimes W$ is a graded $\mathrm{FI}_{\mathcal{W}}$–module of finite type. Over characteristic zero, if U and W have slopes $\leq m$, then $U \otimes W$ will have slope $\leq m$.

Definition 2.22. ($\mathrm{FI}_{\mathcal{W}}$–algebras; $\mathrm{FI}_{\mathcal{W}}$–ideals; The free associative algebra on V). A *(graded) $\mathrm{FI}_{\mathcal{W}}$–algebra* $A = \bigoplus A^i$ is a functor from $\mathrm{FI}_{\mathcal{W}}$ to the category of (graded) k–algebras. A sub–$\mathrm{FI}_{\mathcal{W}}$–module V *generates* A as an $\mathrm{FI}_{\mathcal{W}}$–algebra if V_n generates A_n as a k–algebra for all n.

Let A be a graded $\mathrm{FI}_{\mathcal{W}}$–algebra. An $\mathrm{FI}_{\mathcal{W}}$–ideal I of A is a graded sub–$\mathrm{FI}_{\mathcal{W}}$–algebra of A such that I_n is a homogeneous ideal in A_n for each n.

We define the *free associative algebra* on an $\mathrm{FI}_{\mathcal{W}}$–module V as the graded $\mathrm{FI}_{\mathcal{W}}$–algebra

$$k\langle V \rangle := \bigoplus_{j=0}^{\infty} V^{\otimes j}.$$

Any $\mathrm{FI}_{\mathcal{W}}$–algebra A generated by V admits a surjection of $\mathrm{FI}_{\mathcal{W}}$–algebras $k\langle V \rangle \twoheadrightarrow A$.

If V is supported in nonnegative degrees and has finite type, then so does A. Over a field of characteristic zero, if V has slope $\leq m$ then so does A.

Proposition 2.23. [Wil14, Proposition 5.12] *Let A be an $FI_{\mathcal{W}}$–algebra generated by a graded $FI_{\mathcal{W}}$–module V concentrated in grade d. If V is finitely generated in degree $\leq m$, then the i^{th} graded piece A^i is finitely generated in degree $\leq \left(\frac{i}{d}\right) m$, and moreover if k is a characteristic zero field then weight$(A^i) \leq \left(\frac{i}{d}\right)$weight$(V)$.*

Definition 2.24. (Co–$\mathbf{FI}_{\mathcal{W}}$–modules, Co–$\mathbf{FI}_{\mathcal{W}}$–algebras, finite type). A *graded co–$FI_{\mathcal{W}}$–module* is a functor from the dual category $\mathrm{FI}_{\mathcal{W}}^{op}$ to the category of graded k–modules. A *graded co–$FI_{\mathcal{W}}$–module* is a functor from $\mathrm{FI}_{\mathcal{W}}^{\mathrm{op}}$ to graded k–algebras.

A graded co–$\mathrm{FI}_{\mathcal{W}}$–module V over a field k has *finite type* if its *dual V^**,

$$V_n^* := \mathrm{Hom}_k(V_n, k),$$

has finite type. Similarly, V has *slope* $\leq m$ if V^* does.

Proposition 2.25. [Wil14, Proposition 5.15] **(Finite type co–$\mathbf{FI}_{\mathcal{W}}$–modules generate co–$\mathbf{FI}_{\mathcal{W}}$–algebras of finite type).** *Let k be a Noetherian commutative ring. Suppose that A is a graded co–$FI_{\mathcal{W}}$–algebra containing a graded co–$FI_{\mathcal{W}}$–module V supported in positive grades. If V has finite type, then the subalgebra B of A generated by V is a graded co–$FI_{\mathcal{W}}$–algebra of finite type. When k is a field of characteristic zero and V is a graded co–$FI_{\mathcal{W}}$–module of slope $\leq m$, then B has slope $\leq m$.*

This concludes the summary of the vocabulary and basic results from [Wil14].

3 $\mathbf{FI}_{\mathcal{W}}\sharp$–modules

Church–Ellenberg–Farb [CEF12, Definition 2.19] define FI$\sharp$–modules, a class of sequences of S_n–representations which carry compatible FI_A and co–FI_A–module structures. We will give several characterizations of the analogous constructions in type B/C. An $\mathrm{FI}_{BC}\sharp$–module structure imposes strong constraints on a sequence of B_n–representations; just as for $\mathrm{FI}_A\sharp$–modules [CEF12, Theorem 2.24]: the underlying FI_{BC}–module structure on these sequences must be of the form $\bigoplus_r M_{BC}(U_r)$ for some set of B_r–representations U_r.

Definition 3.1. For $n \geq 0 \in \mathbb{Z}$, let $\mathbf{n_0} := \{0, \pm 1, \pm 2, \ldots, \pm n\}$. We think of the digit 0 as a basepoint. Define $\mathrm{FI}_{BC}\sharp$ to be the category with objects $\mathbf{n_0}$ for $n \geq 0 \in \mathbb{Z}$, and morphisms

$$f : \mathbf{m_0} \to \mathbf{n_0} \qquad \text{such that} \qquad f(-a) = -f(a) \text{ for all } a \in \mathbf{n_0}$$
$$\text{and} \qquad |f^{-1}(b)| \leq 1 \text{ for } 1 \leq |b| \leq n.$$

These morphisms are "injective away from zero". Note that the conditions imply $f(0) = 0$.

We define $\mathrm{FI}_A\sharp$ to be the subcategory with the same objects and morphisms preserving signs. In both cases, an *$\mathrm{FI}_\mathcal{W}\sharp$–module* over a ring k is a functor from $\mathrm{FI}_\mathcal{W}$ to the category of k–modules.

In both types A and B/C, the injective maps in $\mathrm{FI}_\mathcal{W}\sharp$ are precisely the $\mathrm{FI}_\mathcal{W}$ morphisms. We call $f \in \mathrm{Hom}_{\mathrm{FI}_\mathcal{W}\sharp}(\mathbf{m_0}, \mathbf{n_0})$ a *projection* if $|f^{-1}(\pm b)| = 1$ for $1 \leq b \leq n$; these projections are left inverses to the $\mathrm{FI}_\mathcal{W}$ morphisms.

For a morphism $f : \mathbf{m_0} \to \mathbf{n_0}$, we call $|f^{-1}(\{\pm 1, \ldots, \pm n\})|$ the *rank* of f.

This description of $\mathrm{FI}_\mathcal{W}\sharp$–modules was suggested to us by Peter May. The category $\mathrm{FI}_A\sharp$ appears independently in work by May and Merling studying the Segal equivariant infinite loop space machine [MM12], where the category is denoted Π.

Remark 3.2. (A Category $\mathrm{FI}_D\sharp$?). Unlike with FI_A and FI_{BC}, we cannot introduce partial inverses to the morphisms in the category FI_D without also introducing additional automorphisms – and, in fact, generating the entire category $\mathrm{FI}_{BC}\sharp$. It is not clear that we can create any satisfactory analogue of $\mathrm{FI}\sharp$–module theory in type D, since critical properties fail: the FI_D–module structure on $M_D(\mathbf{m})$ does not extend to an $\mathrm{FI}_{BC}\sharp$–module structure.

An alternate characterizations of the categories $\mathrm{FI}_\mathcal{W}\sharp$

In this section we relate the categories $\mathrm{FI}_\mathcal{W}\sharp$ back to the definition of $\mathrm{FI}\sharp$ given by Church–Ellenberg–Farb [CEF12, Definition 2.19].

Remark 3.3. Church–Ellenberg–Farb [CEF12, Definition 2.19] defined $\mathrm{FI}_A\sharp$ to be the category whose objects are finite sets, in which $\mathrm{Hom}_{\mathrm{FI}_A\sharp}(S, T)$ is the set of triples (A, B, ϕ) with A a subset of S, B a subset of T and $\phi : A \to B$ an isomorphism. The composition of two morphisms is given by composition of functions, where the domain is the largest set on which the composition is defined, and the codomain is its bijective image.

We can generalize this definition. We call a subset $A \subseteq \mathbf{n}$ *symmetric* if $a \in A \iff -a \in A$.

Then we define $\mathrm{FI}_{BC}\sharp$ to be the category whose objects are the finite sets $\mathbf{n} = \{\pm 1, \pm 2, \ldots, \pm n\}$, and whose morphisms $\mathrm{Hom}(\mathbf{m}, \mathbf{n})$ are triples (A, B, ϕ) such that A is a symmetric subset of $\mathbf{m}$, B is a symmetric subset of $\mathbf{n}$, and $\phi : A \to B$ is an injective map satisfying $\phi(-a) = -\phi(a)$ for every $a \in A$.

The subcategory $\mathrm{FI}_A\sharp$ comprises all morphisms that preserve signs; this coincides with the definition of $\mathrm{FI}_A\sharp$ given in [CEF12, Definition 2.19].

These definitions of the categories $\text{FI}_A\sharp$ and $\text{FI}_{BC}\sharp$ are equivalent to Definition 3.1. We identify the morphism $(A, B, \phi) \in \text{Hom}_{\text{FI}_{\mathcal{W}}\sharp}(\mathbf{m}, \mathbf{n})$ with the map $f : \mathbf{m_0} \to \mathbf{n_0}$ defined by

$$f : \mathbf{m_0} \to \mathbf{n_0}$$
$$j \mapsto \begin{cases} \phi(j) & j \in A, \\ 0 & j \notin A.. \end{cases}$$

Conversely, we can identify $f : \mathbf{m_0} \to \mathbf{n_0}$ with a triple (A, B, ϕ) by taking $A = f^{-1}(\{\pm 1, \ldots, \pm n\})$, $B = f(A)$, and $\phi = f|_A$. One can check that these identifications of morphisms are consistent with the composition rules, and give an isomorphism of categories.

Examples of $\text{FI}_{\mathcal{W}}\sharp$–modules

We prove in Proposition 3.4 and Corollary 3.5 that FI_{BC}–modules of the form $M_{BC}(\mathbf{m})$ or $M_{BC}(U)$ have $\text{FI}_{BC}\sharp$–module structures.

Proposition 3.4. ($M_{\mathcal{W}}(\mathbf{a})$ is an $\text{FI}_{\mathcal{W}}\sharp$–module). *Let $\mathcal{W}_n$ be S_n or B_n. For $a \geq 0$, the $FI_{\mathcal{W}}$–module structure on $M_{\mathcal{W}}(\mathbf{a})$ extends to an $FI_{\mathcal{W}}\sharp$–structure.*

Proof of Proposition 3.4. By Definition 2.3, $M_{\mathcal{W}}(\mathbf{a})_m = \text{Span}_k \{e_s \mid s \in \text{Hom}_{\text{FI}_{\mathcal{W}}}(\mathbf{a}, \mathbf{m})\}$.

Take any $\text{FI}_{\mathcal{W}}\sharp$–morphism $f : \mathbf{m_0} \to \mathbf{n_0}$. We define

$$f \cdot e_s = \begin{cases} e_{f \circ s} & \text{if } 0 \notin f(s(\mathbf{a})) \,, \\ 0 & \text{otherwise.} \end{cases}$$

The condition $0 \notin f(s(\mathbf{a}))$ is the statement that $f \circ s$ is an injective map $\mathbf{a} \to \mathbf{n}$. Given $g : \mathbf{n_0} \to \mathbf{p_0}$, we note that

$$0 \in \Big(g \circ f\Big)\Big(s(\mathbf{a})\Big) \quad \Longleftrightarrow \quad 0 \in g\Big((f \circ s)(\mathbf{a})\Big);$$

this implies that the action $(g \circ f) \cdot e_s = g \cdot (f \cdot e_s)$ is functorial. □

Corollary 3.5. ($M_{\mathcal{W}}(U)$ is an $\text{FI}_{\mathcal{W}}\sharp$–module). *Let $\mathcal{W}_n$ be S_n or B_n. Given a $\mathcal{W}_a$–representation U, the $FI_{\mathcal{W}}$–module*

$$M_{\mathcal{W}}(U) := M_{\mathcal{W}}(\mathbf{a}) \otimes_{k[\mathcal{W}_a]} U$$

has the structure of an $FI_{\mathcal{W}}\sharp$–module.

Remark 3.6. We note the proof of Proposition 3.4 does not work in type D, as the space $M_D(\mathbf{m})_m \subseteq M_{BC}(\mathbf{m})_m$ is not closed under the action of action of the $\text{FI}_{BC}\sharp$–morphisms.

3.1 Classification of $\text{FI}_{BC}\sharp$–modules

The structure of an $\text{FI}_{BC}\sharp$–module is highly constrained. In Corollary 3.5 we saw that $M_{BC}(U)$ is an $\text{FI}_{BC}\sharp$–module. Just as Church–Ellenberg–Farb proved with $\text{FI}_A\sharp$–modules [CEF12, Theorem 2.24], we will now find that all $\text{FI}_{BC}\sharp$–modules are sums of $\text{FI}_{BC}\sharp$–modules of this form. For the following theorem, recall the definition of $\text{FB}_{\mathcal{W}}$–Mod from Definition 2.3.

Theorem 3.7. **($\text{FI}_{\mathcal{W}}\sharp$–modules take the form $\bigoplus_{a=0}^{\infty} M_{\mathcal{W}}(U_a)$).** *Let $\mathcal{W}_n$ be S_n or B_n. Every $\text{FI}_{\mathcal{W}}\sharp$–module V is of the form*

$$V = \bigoplus_{a=0}^{\infty} M_{\mathcal{W}}(U_a), \qquad U_a \text{ a representation of } \mathcal{W}_a \text{ (possibly } U_a = 0\text{)},$$

and moreover that the maps

$$M_{\mathcal{W}}(-) : FB_{\mathcal{W}}\text{–}Mod \longrightarrow FI_{\mathcal{W}}\text{–}Mod \qquad \text{and} \qquad H_0(-) : FI_{\mathcal{W}}\text{–}Mod \longrightarrow FB_{\mathcal{W}}\text{–}Mod$$

define an equivalence of categories.

We remark that Theorem 3.7, and in particular the result in type A proven by Church–Ellenberg-Farb [CEF12, Theorem 2.24], parallels earlier work of Pirashvili [Pir00a, Pir00b] on Γ*–modules*, functors from the category of finite based sets and (not necessarily injective) based maps.

This theorem is proved for FI_A in [CEF12, Theorem 2.24], and their proof adapts readily to the general case.

Church–Ellenberg–Farb proceed by induction. Assume V is an $\text{FI}_A\sharp$–module, and fix n such that $V_m = 0$ for all $m < n$ (possibly $n = 1$). They define a particular idempotent endomorphism of $\text{FI}_A\sharp$–modules $E : V \to V$, and prove the resultant decomposition

$$V = EV \oplus \ker(E) \cong M_A(V_n) \oplus \ker(E).$$

Their same proof carries through exactly in the case $\text{FI}_{BC}\sharp$–modules if we redefine the endomorphism E as follows, for $m \geq n$,

$$\begin{aligned} E_m &: V_m \to V_m \\ E_m &= \sum_{\substack{S \subseteq \mathbf{m},\ |S|=n \\ S \text{ symmetric}}} I_S \qquad \text{where} \qquad I_S(j) = \begin{cases} j & \text{if } j \in S\,, \\ 0 & \text{otherwise} \end{cases} \\ &\qquad\qquad\qquad\qquad \in \text{Hom}_{\text{FI}_{BC}\sharp}(\mathbf{m_0}, \mathbf{m_0}). \end{aligned}$$

In the notation of Remark 3.3, $I_S = (S, S, \text{identity})$. Again we conclude

$$V \cong M_{BC}(V_n) \oplus \ker(E)$$

with $\ker(E)$ vanishing in degree n, and the desired decomposition follows by induction on n.

Church–Ellenberg–Farb further argue that, since maps $F : V \to V'$ of $\mathrm{FI}_A\sharp$–modules commute with E and preserves this decomposition, the map $M_A(V_n) \to M_A(V_n')$ must be induced by some map of S_n–representations $V_n \to V_n'$. Their arguments hold for FI_{BC}–modules, and imply the equivalence of the categories FB_{BC}–Mod and $\mathrm{FI}_{BC}\sharp$–Mod.

Corollary 3.8. *Let $\mathcal{W}_n$ be S_n or B_n. With V an $\mathrm{FI}_\mathcal{W}\sharp$–module as above, any sub–$\mathrm{FI}_\mathcal{W}\sharp$–module of V is of the form $\bigoplus_{a=0}^{\infty} M_\mathcal{W}(U_a')$ for some $\mathcal{W}_a$–representations $U_a' \subseteq U_a$ (possibly zero).*

Corollary 3.9. *Let $\mathcal{W}_n$ be S_n or B_n. If V is an $\mathrm{FI}_\mathcal{W}$–module generated in degree $\leq d$, then any sub–$\mathrm{FI}_\mathcal{W}\sharp$–module of V is also generated in degree $\leq d$.*

Corollary 3.10. *Let $\mathcal{W}_n$ be S_n or B_n. An $\mathrm{FI}_\mathcal{W}\sharp$–module V has injectivity degree 0. If V is generated in degree $\leq d$, then V has stability degree $\leq d$, as do its sub–$\mathrm{FI}_\mathcal{W}$–submodules.*

Proof of Corollary 3.10. Let U_m be any $\mathcal{W}_m$–representation. In [Wil14, Example 3.14] we saw that $M_\mathcal{W}(U_m)$ is generated in degree m. In [Wil14, Proposition 4.17] we further saw that $M_\mathcal{W}(U_m)$ has injectivity degree 0 and surjectivity degree m. By assumption that V is generated in degree $\leq d$, we may decompose V into a sum $V = \bigoplus_{m=0}^{d} M_\mathcal{W}(U_m)$ and the result follows. □

Corollary 3.11. *If V is an $\mathrm{FI}_{BC}\sharp$–module over characteristic zero, generated in degree d. Then $\{V_n\}$ is uniformly representation stable in degree $\leq 2d$.*

Proof of Corollary 3.11. Any such $\mathrm{FI}_{BC}\sharp$–module has weight $\leq d$ by Theorem 2.12, and stability degree $\leq d$ by Corollary 3.10. The conclusion follows from Theorem 2.16. □

Corollary 3.12. *Let $\mathcal{W}_n$ be S_n or B_n. An $\mathrm{FI}_\mathcal{W}\sharp$–module V is completely determined by the sequence of $\mathcal{W}_n$–representations $\{V_n\}$, equivalently (over characteristic zero) by the sequence of characters $\{\chi_n\}$.*

Proof of Corollary 3.12. We can construct $H_0(V)$ inductively from the sequence $\{V_n\}$ of $\mathcal{W}_n$–representations:

$$H_0(V)_0 = V_0 \qquad \text{and} \qquad H_0(V)_n = V_n \;/\; \text{span} \coprod_{k<n} M_\mathcal{W}\Big(H_0(V)_k\Big)_n$$

The $\mathrm{FI}_\mathcal{W}\sharp$–module structure on V is determined by the identification $V \cong M_\mathcal{W}(H_0(V))$. □

Corollary 3.13. *If k is a field, and V an $FI_{BC}\sharp$–module k. Then*

$$V \text{ is finitely generated in degree } \leq d \iff \dim_k(V_n) = O(n^d)$$
$$\iff \dim_k(V_n) = P(n) \text{ for a polynomial } P \in \mathbb{Q}[T] \text{ of degree at most } d$$

If k is a commutative ring, then an $FI_{BC}\sharp$–module V over k is finitely generated in degree $\leq d$ if and only if V_n is generated by $O(n^d)$ elements.

Proof of Corollary 3.13. The statements follow from Theorem 3.7 and the same argument used to prove [CEF12, Corollary 2.27]. In type B/C, the polynomial P is determined by the formula

$$\dim_k M_{BC}(U)_n = \binom{n}{m} \dim_k U \qquad \text{for a } B_m\text{–representation } U. \quad \square$$

4 The character polynomials

4.1 Character polynomials for the symmetric groups

A major result of Church–Ellenberg–Farb is that, given a finitely generated FI_A–module over a field of characteristic zero, the characters of the S_n–representations V_n have a particularly nice form. They are, for n sufficiently large, given by a character polynomial (independent of n), as we now define.

Definition 4.1. (Character Polynomials for S_n). Let k be a characteristic zero field. For $r \geq 1$ and $n \geq 0$, let X_r be the class function on S_n defined by

$$X_r(s) := \text{ the number of } r\text{–cycles in the cycle type of } s.$$

For fixed n, the monomials in the functions $X_1, \ldots, X_n$ span the space of class functions on S_n, subject to some relations – for example, relations imposed by the fact that any element's cycle lengths sum to n. As functions on the disjoint union $\coprod_{n=0}^{\infty} S_n$, however, the functions X_r are algebraically independent, and define a polynomial ring $k[X_1, X_2, \ldots,]$. We call elements of this ring the *character polynomials* of the symmetric groups, and define the total degree of a character polynomial by assigning $\deg(X_r) = r$.

Theorem 4.2. *[CEF12, Theorem 2.67]* **(Polynomiality of characters for S_n).** *Let k be a field of characteristic zero, and let V be a finitely generated FI_A–module with weight $\leq d$ and stability degree $\leq s$. There exists a unique polynomial $P_V \in k[X_1, X_2, \ldots]$ such that*

$$\chi_{V_n}(\sigma) = P_V(\sigma) \qquad \text{for all } n \geq s + d \text{ and all } \sigma \in S_n.$$

The polynomial P_V has degree at most d. By setting $F_V(n) = P_V(n, 0, ..., 0)$ we have:

$$dim_k(V_n) = \chi_{V_n}(id) = F_V(n) \qquad \text{for all } n \geq s + d.$$

If V is an $FI_A\sharp$–module then the above equalities hold for all $n \geq 0$.

Background and formulas for character polynomials of the symmetric groups. Church–Ellenberg–Farb [CEF12] prove these theorems using the classical result that the character of the irreducible representations $V(\lambda)_n$, written here in the notation defined in Section 2.1, is given by a character polynomial P^λ independent of n. These character polynomials were described by Murnaghan in 1951 [Mur51] and by Specht in 1960 [Spe60]. This independence of the characters from n was known to Murnaghan in 1937 [Mur37].

Formulas for the character polynomial P^λ associated to the irreducible representations $V(\lambda)_n$ are given in Macdonald's book [Mac79]. In 2009, new formulas were published by Garsia and Goupil [GG09], which they used to study the combinatorics of Kronecker coefficients. To state these formulas, we use the following notation:

Let λ be a partition of n. We define the *length* of λ, $\ell(\lambda) :=$ the number of parts of λ.
For $r \geq 1$, we write $n_r(\lambda) :=$ the number of parts of λ of length r.

We further define the integer z_λ so that $\frac{n!}{z_\lambda}$ is the number of elements in S_n of cycle type λ. Explicitly,

$$z_\lambda = \prod_r^n r^{n_r(\lambda)} n_r(\lambda)!$$

We write χ^λ to mean the character of the irreducible S_n–representation V_λ. We define $\chi^\varnothing := 1$. If χ is any class function on S_n, and ρ a partition of n, then we write χ_ρ to mean the value of χ on elements of cycle type ρ.

Definition 4.3. (Generalized Binomial Coefficients). Let ρ be a partition of m. Following Macdonald [Mac79, I.7.13(a)], we define *generalized binomial coefficients*:

$$\binom{\mathbf{X}}{\rho} := \prod_r \binom{X_r}{n_r(\rho)} = \prod_r \frac{X_r(X_r - 1)\cdots(X_r - n_r(\rho) + 1)}{n_r(\rho)!},$$

For example,

$$\binom{\mathbf{X}}{(3,2,2,1)} := \binom{X_3}{1}\binom{X_2}{2}\binom{X_1}{1} = X_3 \frac{X_2(X_2 - 1)}{2} X_1 = \frac{1}{2} X_3 X_2^2 X_1 - \frac{1}{2} X_3 X_2 X_1$$

Remark 4.4. (Indicator Functions for the Conjugacy Classes of S_m) Given a partition λ of m, and $s \in S_m$, note that the generalized binomial coefficient

$$\binom{\mathbf{X}}{\lambda}(s) = \begin{cases} 1 & \text{if } s \text{ has cycle type } \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

This polynomial's restriction to $S_m \subseteq \coprod_{n\geq 0} S_n$ is an indicator function for the conjugacy class of cycle type λ. Polynomials of this form give a convenient basis for $k[X_1, X_2, \ldots,]$.

Since the binomial coefficient in an indeterminate X

$$\binom{X}{m} = \frac{X(X-1)(X-2)\cdots(X-m+1)}{m!}$$

is a polynomial in X of degree m, the generalized binomial coefficient $\binom{\mathbf{X}}{\lambda}$ is a polynomial of total degree $\sum r \cdot n_r(\lambda) = m$ in $k[X_1, X_2, \ldots]$.

Proposition 4.5. *([Mac79, I.7.14])*

For $\lambda \vdash m$, a formula for the character P^λ of the irreducible S_n–representation $V(\lambda)_n$ is given as follows:

$$P^\lambda = \sum_{\substack{\text{Partitions } \rho,\sigma \\ |\rho|+|\sigma|=|\lambda|}} \frac{(-1)^{\ell(\sigma)}\ \chi^\lambda_{(\rho\cup\sigma)}}{z_\sigma} \binom{\mathbf{X}}{\rho}.$$

By Remark 4.4, this is a character polynomial of degree $|\lambda| = m$.

4.2 Character polynomials in type B/C and D

We can analogously define character polynomials for the hyperoctahedral group B_n and the even-signed permutation groups D_n. Our goal is to prove that the characters of a finitely generated $\mathrm{FI}_{\mathcal{W}}$–module are equal (for n sufficiently large) to a character polynomial. A summary of the main result:

Theorem 4.6. (Summary: Finitely generated $\mathbf{FI}_{\mathcal{W}}$–modules have character polynomials). *Let V be an $\mathrm{FI}_{\mathcal{W}}$–module over characteristic zero finitely generated in degree $\leq d$. Let χ_{V_n} denote the character of the $\mathcal{W}_n$–representation V_n. Then there exists a unique character polynomial F_V of degree at most d such that $F_V(\sigma) = \chi_{V_n}(\sigma)$ for all $\sigma \in \mathcal{W}_n$, for all n sufficiently large.*

The full statement of this result in type B/C and D, including bounds on the stable range, is given in Theorem 4.16 in Section 4.2. To this end, we will first develop the theory in type B/C, and from there we can use our methods of inducing FI_D–modules to FI_{BC} to recover results in type D.

Recall from Section 2.1.2 that the conjugacy classes for B_n are classified by double partitions (α, β) of n, designating the signed cycle type of each element. Given a character (or class function) χ of a B_n–representation, we will write $\chi_{(\alpha,\beta)}$ to denote the value of χ on elements of signed cycle type (α, β).

Definition 4.7. (Character Polynomials for B_n). For $r \geq 1$ and $n \geq 0$, let X_r and Y_r be the class functions on B_n defined by

$$X_r(w) = \text{ the number of positive } r\text{–cycles in the cycle type of } w.$$

$$Y_r(w) = \text{ the number of negative } r\text{–cycles in the cycle type of } w.$$

Again, these functions form a polynomial ring $k[X_1, Y_1, X_2, Y_2, \ldots]$ where we designate $deg(X_r) = deg(Y_r) = r$.

Example 4.8. We saw in Example 1.6 that $V_n = V((n-1),(1)) \cong k^n$, the canonical B_n–representation by signed permutation matrices, has characters $\chi^V = X_1 - Y_1$ for all n. Similarly, one can compute that the characters of $V_n = \bigwedge^2 V\big((n-1),(1)\big)$ are

$$\chi^{\bigwedge^2 V} = \frac{1}{2}X_1(X_1 - 1) + \frac{1}{2}Y_1(Y_1 - 1) - X_1Y_1 - X_2 + Y_2 \qquad \text{for all } n,$$

and that the characters of $V_n = \mathrm{Sym}^2 V\big((n-1),(1)\big)$ are

$$\chi^{\mathrm{Sym}^2 V} = \frac{1}{2}X_1(X_1 + 1) + \frac{1}{2}Y_1(Y_1 + 1) - X_1Y_1 + X_2 - Y_2 \qquad \text{for all } n.$$

Remark 4.9. (Indicator Functions for the Conjugacy Classes of B_m) Given a double partition (λ, ν) of m, and $w \in B_m$, note that the degree m character polynomial

$$\binom{\mathbf{X}}{\lambda}\binom{\mathbf{Y}}{\nu}(w) = \begin{cases} 1 & \text{if } w \text{ has signed cycle type } (\lambda, \nu), \\ 0 & \text{otherwise.} \end{cases}$$

Again $\binom{\mathbf{X}}{\lambda}\binom{\mathbf{Y}}{\nu}$ is a polynomial of degree $\sum r \cdot n_r(\lambda) + \sum r \cdot n_r(\nu) = m$ that is an indicator function on B_m of the signed conjugacy class (λ, ν).

Remark 4.10. (Restricting Characters to $S_n \subseteq B_n$). The symmetric group S_n forms the subgroup of B_n generated by the (necessarily positive) cycles that preserve signs. Thus, if V is a B_n–representations with character χ^V given by some character polynomial $P_V \in k[X_1, Y_1, X_2, Y_2, \ldots]$, the character for $\mathrm{Res}^{B_n}_{S_n} V$ is given by the element in $k[X_1, X_2, \ldots]$ obtained by evaluating each variable Y_r in P_V at 0.

4.2.1 The character of $V(\lambda,\mu)_n$ is independent of n

Recall from Section 2.1.2 that, given a double partition (λ,ν) of d with $\nu \vdash m$, then $V(\lambda,\nu)_n$ denotes the irreducible B_n–representation associated to the double partition $(\lambda[n-m],\nu)$.

Theorem 4.11. (The character of $V(\lambda,\mu)_n$ is independent of n). *If (λ,ν) is a double partition of d, then there is a character polynomial $P^{(\lambda,\nu)}$ of degree at most d equal to the character of the irreducible B_n–representations $V(\lambda,\nu)_n$ for all n.*

Explicitly, $P^{(\lambda,\nu)}$ is given as follows. Let $m=|\nu|$, and define μ so that $\nu=\mu[m]$; for $\nu=\varnothing$ take $\mu=\varnothing$. Then

$$P^{(\lambda,\nu)} = \sum_{\substack{(\alpha,\beta)\\ |\alpha|+|\beta|=|\nu|,}} \; \sum_{\substack{\textit{Partitions } \rho,\sigma\\ |\rho|+|\sigma|=|\mu|}} \; \sum_{\substack{\textit{Partitions } \xi,\eta\\ |\xi|+|\eta|=|\lambda|}} (-1)^{\ell(\beta)} \left(\frac{(-1)^{\ell(\sigma)}\,\chi^{\mu}_{(\rho\cup\sigma)}}{z_\sigma} \right) \left(\frac{(-1)^{\ell(\eta)}\,\chi^{\lambda}_{(\xi\cup\eta)}}{z_\eta} \right)$$
$$\left(\prod_r \binom{n_r(\alpha)+n_r(\beta)}{n_r(\rho)} \binom{X_r - n_r(\alpha) + Y_r - n_r(\beta)}{n_r(\xi)} \binom{X_r}{n_r(\alpha)} \binom{Y_r}{n_r(\beta)} \right).$$

For example,

$$P^{\left(\square\,,\,\square\right)} = (X_1-Y_1)(X_1+Y_1-2) \qquad \text{and} \qquad P^{\left(\varnothing\,,\,\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)} = \binom{X_1}{2} + \binom{Y_1}{2} - X_2 - X_1Y_1 + Y_2$$

We will prove Theorem 4.11 in four steps. Our first step, Lemma 4.12, is to prove the result for representations of the form $V(\lambda,\varnothing)_n$. In the second step, Lemma 4.13, we produce a formula for characters of representations $V(\varnothing,\lambda[n])$. Our third step, Lemma 4.15, is to compute the character of an induced representation of the form $\mathrm{Ind}_{B_m\times B_{n-m}}^{B_n} U \boxtimes U'$, and the final step will be to derive the formula in Theorem 4.11.

Lemma 4.12. (Step 1: The character of $V(\lambda,\varnothing)_n$). *Let λ be a partition of m. Then, for each n, the character of B_n–representation $V(\lambda,\varnothing)_n$ is given by the character polynomial $P^{(\lambda,0)}$*

$$\begin{aligned} P^{(\lambda,0)} &= \sum_{\substack{\textit{Partitions } \rho,\sigma\\ |\rho|+|\sigma|=|\lambda|}} \frac{(-1)^{\ell(\sigma)}\,\chi^{\lambda}_{(\rho\cup\sigma)}}{z_\sigma} \binom{\mathbf{X}+\mathbf{Y}}{\rho} \\ &:= \sum_{\substack{\textit{Partitions } \rho,\sigma\\ |\rho|+|\sigma|=|\lambda|}} \frac{(-1)^{\ell(\sigma)}\,\chi^{\lambda}_{(\rho\cup\sigma)}}{z_\sigma} \prod_r \binom{X_r+Y_r}{n_r(\rho)} \end{aligned}$$

Proof of Lemma 4.12. As described in Section 2.1.2, the B_n–representations $V(\lambda, \varnothing)_n$ are by definition the pullback of the S_n–representation $V(\lambda)_n$ under the natural surjection $B_n \twoheadrightarrow S_n$. This map takes positive and negative r-cycles in B_n to r-cycles in S_n; a signed permutation of signed cycle type (μ, ν) is mapped to a permutation of type $\mu \cup \nu$. It follows that a hyperoctahedral character polynomial for $V(\lambda, \varnothing)_n$ can be obtained from the symmetric character polynomial for $V(\lambda)_n$ by replacing each X_r with the sum $X_r + Y_r$. The formula therefore follows from Macdonald's formula, Proposition 4.5. ☐

Lemma 4.13. (Step 2: The character of $V(\varnothing, \lambda[n])$). *Let n be fixed, and consider a partition $\lambda[n]$ of n. Then the character $\chi^{(\varnothing,\lambda[n])}$ of the B_n–representation $V(\varnothing, \lambda[n])$ takes the following value on B_n elements of signed cycle type (α, β):*

$$\chi^{(\varnothing,\lambda[n])}_{(\alpha,\beta)} = (-1)^{\ell(\beta)} P^{(\lambda,0)}(\alpha, \beta).$$

Remark 4.14. We note that this formula for the character $V(\varnothing, \lambda[n])$ is not a B_n character polynomial, since the coefficient $(-1)^{\ell(\beta)}$ depends on the cycle type (α, β).

Proof of Lemma 4.13. Recall from Section 2.1.2 that $\varepsilon : B_n \to B_n/D_n \cong \{\pm 1\}$ is the character mapping an element $w \in B_n$ to -1 precisely when w reverses an odd number of signs. Since positive cycles reverse an even number of signs, and negative cycles reverse an odd number, the character ε takes the value $(-1)^{\ell(\beta)}$ on elements of signed cycle type (α, β).

By definition,

$$V(\varnothing, \lambda[n]) = V(\lambda[n], \varnothing) \otimes \varepsilon = V(\lambda, \varnothing)_n \otimes \varepsilon$$

and so the formula follows from Lemma 4.13. ☐

Lemma 4.15. (Step 3: The character of $\mathrm{Ind}^{B_n}_{B_m \times B_{n-m}} U \boxtimes U'$). *Suppose that U is a B_m–representation with character χ^U, and that U' is a B_{n-m}–representation, with character $\chi^{U'}$. Then the character $\chi^{(U,U')}$ of the induced B_n–representation $\mathrm{Ind}^{B_n}_{B_m \times B_{n-m}} U \boxtimes U'$ is given by:*

$$\chi^{(U,U')}_{(\rho,\sigma)} = \left(\sum_{\substack{(\alpha,\beta) \\ |\alpha|+|\beta|=m,}} \chi^U_{(\alpha,\beta)} \, \chi^{U'}_{(\delta,\gamma)} \binom{\mathbf{X}}{\alpha} \binom{\mathbf{Y}}{\beta} \right) (\rho, \sigma)$$

where (δ, γ) is the double partition of $(n - m)$ such that $(\rho, \sigma) = (\alpha \cup \delta, \beta \cup \gamma)$. It is well-defined, since $\left(\binom{\mathbf{X}}{\alpha} \binom{\mathbf{Y}}{\beta} \right)(\rho, \sigma)$ will vanish unless such a decomposition of (ρ, σ) exists.

We note that Lemma 4.15 holds when k is $\mathbb{Z}$ or any field.

Proof of Lemma 4.15. Let $w \in (B_m \times B_{n-m})$, and let p_m and p_{n-m} denote the projections of w onto B_m and B_{n-m}, respectively. The character of the $(B_m \times B_{n-m})$–representation $U \boxtimes U'$ is

$$\chi^{U \boxtimes U'} = \chi^U(p_m(w)) \cdot \chi^{U'}(p_{n-m}(w)).$$

The character of the induced representation $\mathrm{Ind}_{B_m \times B_{n-m}}^{B_n} U \boxtimes U'$ is

$$\begin{aligned}\chi^{(U,U')}(w) &= \sum_{\substack{\{\text{cosets } C \mid w \cdot C = C\} \\ \text{any } s \in C}} \chi^{U \boxtimes U'}(s^{-1}ws) \\ &= \sum_{\substack{\{\text{cosets } C \mid w \cdot C = C\} \\ \text{any } s \in C}} \chi^U(p_m(s^{-1}ws)) \cdot \chi^{U'}(p_{n-m}(s^{-1}ws))\end{aligned}$$

summed over all cosets C in $B_n/(B_m \times B_{n-m})$ that are stabilized by w, equivalently, those cosets C such that $s^{-1}ws \in (B_m \times B_{n-m})$ for any $s \in C$.

The cosets $B_n/(B_m \times B_{n-m})$ correspond to the orbit of the sets

$$\{\{-1,1\},\dots,\{-m,m\}\} \qquad \text{and} \qquad \{\{-(m+1),(m+1)\},\dots,\{-n,n\}\}$$

under the action of B_n; they are indexed by all partitions of $\{\{-1,1\},\{-2,2\},\dots,\{-n,n\}\}$ into a set of m blocks and a set of $(n-m)$ blocks.

An element $w \in B_n$ can be conjugated into $(B_m \times B_{n-m})$ precisely when its positive and negative cycles can be partitioned into a set of cycles of total length m, and a set of cycles of total length $(n-m)$. If we fix a double partition (α, β) of m, then the cycles of w can be factored into an element w_m of cycle type (α, β) and its complement w_{n-m} in the following number of ways (possibly 0):

$$\binom{\mathbf{X}}{\alpha}\binom{\mathbf{Y}}{\beta}(w) := \binom{X_1(w)}{n_1(\alpha)}\binom{X_2(w)}{n_2(\alpha)}\cdots\binom{X_m(w)}{n_m(\alpha)}\binom{Y_1(w)}{n_1(\beta)}\binom{Y_2(w)}{n_2(\beta)}\cdots\binom{Y_m(w)}{n_m(\beta)}.$$

Each such factorization of w corresponds to a coset $C \in B_n/(B_m \times B_{n-m})$ that is stabilized by w. For any representative $s \in C$, $p_m(s^{-1}ws)$ has signed cycle type (α, β).

Thus, if we denote the signed cycle type of w_{n-m} by (δ, γ), we conclude

$$\chi^{(U,U')}(w) = \left(\sum_{\substack{(\alpha,\beta) \\ |\alpha|+|\beta|=m,}} \chi^U_{(\alpha,\beta)}\, \chi^{U'}_{(\delta,\gamma)} \binom{\mathbf{X}}{\alpha}\binom{\mathbf{Y}}{\beta} \right)(w). \quad \square$$

Proof of Theorem 4.11. **(Step 4: The Character of $V(\lambda,\nu)_n$).** Let (λ,ν) be a double partition of d, with $|\nu| = m$ and $|\lambda| = (d-m)$. From the construction of the irreducible representations of B_n described in Section 2.1.2,

$$V(\lambda,\nu)_n = \mathrm{Ind}_{B_{n-m}\times B_m}^{B_n} V(\lambda,\varnothing)_{n-m} \boxtimes V(\varnothing,\nu).$$

We wish to compute a character polynomial $P^{(\lambda,\nu)}$ which gives the character for $V(\lambda,\nu)_n$ for each n.

By Lemma 4.15,

$$\chi^{(\lambda[n],\nu)}(w) = \left(\sum_{\substack{(\alpha,\beta)\\ |\alpha|+|\beta|=|\nu|,}} \chi^{(\varnothing,\nu)}_{(\alpha,\beta)}\, \chi^{(\lambda[n-m],\varnothing)}_{(\delta,\gamma)} \binom{\mathbf{X}}{\alpha}\binom{\mathbf{Y}}{\beta}\right)(w)$$

with (δ,γ) the double partition of $(n-m)$ such that $(\alpha\cup\delta, \beta\cup\gamma)$ is the signed cycle type of w.

We write $\nu = \mu[m]$, where μ is the partition obtained from ν by discarding the largest part; thus, by Lemmas 4.13 and 4.12,

$$\begin{aligned}
\chi^{(\varnothing,\mu[m])}_{(\alpha,\beta)} &= (-1)^{\ell(\beta)} P^{(\mu,0)}(\alpha,\beta)\\
&= (-1)^{\ell(\beta)} \sum_{\substack{\text{Partitions } \rho,\sigma\\ |\rho|+|\sigma|=|\mu|}} \frac{(-1)^{\ell(\sigma)}\ \chi^{\mu}(\rho\cup\sigma)}{z_\sigma} \binom{\mathbf{X}+\mathbf{Y}}{\rho}(\alpha,\beta)\\
&= (-1)^{\ell(\beta)} \sum_{\substack{\text{Partitions } \rho,\sigma\\ |\rho|+|\sigma|=|\mu|}} \frac{(-1)^{\ell(\sigma)}\ \chi^{\mu}(\rho\cup\sigma)}{z_\sigma} \prod_r \binom{n_r(\alpha)+n_r(\beta)}{n_r(\rho)}
\end{aligned}$$

Moreover, since for each r we have

$$n_r(\delta) = X_r(w) - n_r(\alpha) \qquad \text{and} \qquad n_r(\gamma) = Y_r(w) - n_r(\beta),$$

we can use Lemma 4.12 to compute:

$$\begin{aligned}
\chi^{(\lambda[n-m],\varnothing)}_{(\delta,\gamma)} &= P^{(\lambda,0)}(\delta,\gamma)\\
&= \sum_{\substack{\text{Partitions } \xi,\eta\\ |\xi|+|\eta|=|\lambda|}} \frac{(-1)^{\ell(\eta)}\ \chi^{\lambda}(\xi\cup\eta)}{z_\eta} \prod_r \binom{X_r+Y_r}{n_r(\xi)}(\delta,\gamma)\\
&= \sum_{\substack{\text{Partitions } \xi,\eta\\ |\xi|+|\eta|=|\lambda|}} \frac{(-1)^{\ell(\eta)}\ \chi^{\lambda}(\xi\cup\eta)}{z_\eta} \prod_r \binom{X_r - n_r(\alpha) + Y_r - n_r(\beta)}{n_r(\xi)}(w)
\end{aligned}$$

Putting these together,

$$\begin{aligned}
\chi^{(\lambda[n-m],\nu)}(w) &= \left(\sum_{\substack{(\alpha,\beta) \\ |\alpha|+|\beta|=|\nu|,}} \chi^{(\varnothing,\nu)}_{(\alpha,\beta)} \, \chi^{(\lambda[n-m],\varnothing)}_{(\delta,\gamma)} \binom{\mathbf{X}}{\alpha}\binom{\mathbf{Y}}{\beta} \right)(w) \\
&= \left(\sum_{\substack{(\alpha,\beta) \\ |\alpha|+|\beta|=|\nu|,}} \binom{\mathbf{X}}{\alpha}\binom{\mathbf{Y}}{\beta} \right) (-1)^{\ell(\beta)} \left(\sum_{\substack{\text{Partitions } \rho,\sigma \\ |\rho|+|\sigma|=|\mu|}} \frac{(-1)^{\ell(\sigma)} \, \chi^{\mu}(\rho \cup \sigma)}{z_\sigma} \prod_r \binom{n_r(\alpha)+n_r(\beta)}{n_r(\rho)} \right) \\
&\quad \left(\sum_{\substack{\text{Partitions } \xi,\eta \\ |\xi|+|\eta|=|\lambda|}} \frac{(-1)^{\ell(\eta)} \, \chi^{\lambda}(\xi \cup \eta)}{z_\eta} \prod_r \binom{X_r - n_r(\alpha) + Y_r - n_r(\beta)}{n_r(\xi)} \right)(w)
\end{aligned}$$

which gives the desired formula.

Note that the degree of $P^{(\lambda,\nu)}$

$$\begin{aligned}
\deg(P^{(\lambda,\nu)}) &\leq \left(|\alpha| + |\beta| + \max_{\substack{\text{Partitions } \xi,\eta \\ |\xi|+|\eta|=|\lambda|}} |\xi| \right) \\
&= (|\nu| + |\lambda|) \\
&= d
\end{aligned}$$

so $\deg(P^{(\lambda,\nu)})$ is at most the size of the double partition (λ,ν), as claimed. □

4.3 Finite generation and character polynomials

We can now use Theorem 4.11 to prove the existence of character polynomials for finitely generated FI_{BC}–modules in Theorem 4.16. As a consequence of Theorems 4.2 and 4.16, we can determine a number of constraints on the structure of finitely generated $\mathrm{FI}_{\mathcal{W}}$–modules.

Theorem 4.16. (Characters of finitely generated $\mathrm{FI}_{\mathcal{W}}$–modules are eventually polynomial). *Let k be a field of characteristic zero. Suppose that V is a finitely generated FI_{BC}–module with weight $\leq d$ and stability degree $\leq s$, or, alternatively, suppose that V is a finitely generated FI_D–module with weight $\leq d$ such that $Ind_D^{BC}\, V$ has stability degree $\leq s$. In either case, there is a unique polynomial*

$$F_V \in k[X_1, Y_1, X_2, Y_2, \dots],$$

independent of n, such that the character of $\mathcal{W}_n$ on V_n is given by F_V for all $n \geq s+d$. The polynomial F_V has degree $\leq d$, with $deg(X_i) = deg(Y_i) = i$.

We remark that, by Theorem 2.12, d is at most the degree of generation of V.

Proof of Theorem 4.16. Assume first that V is a finitely generated FI_{BC}–module.

Recall that the class functions X_i, Y_i are algebraically independent as functions on the disjoint union of all hyperoctahedral groups $\coprod_{n\geq 0} B_n$. The uniqueness of a character polynomial F_V therefore follows from the general fact that any two (multivariate) polynomials that agree on all but finitely many integer points are necessarily equal. We turn to proving existence of the character polynomial F_V.

By Theorem 2.16, for $n \geq s+d$, V_n has a decomposition

$$V_n = \bigoplus_{\lambda} c_\lambda V(\lambda)_n$$

where by assumption c_λ is only nonzero for $|\lambda| \leq d$. Thus for $n \geq s+d$ the characters V_n are given by a character polynomial of degree $\leq d$ by Theorem 4.11.

We will now use this result to prove the theorem for type D. That V is an FI_D–module of weight $\leq d$ means by definition that $\mathrm{Ind}_D^{BC} V$ is an FI_{BC}–module of weight $\leq d$, and $\mathrm{Ind}_D^{BC} V$ moreover has stability degree $\leq s$ by assumption. Hence the B_n–representations $(\mathrm{Ind}_D^{BC} V)_n$ are given by a unique character polynomial F_V for all $n \geq s+d$. Moreover, if V is generated in degree $\leq m$, then

$$V_n \cong (\mathrm{Res}_D^{BC}\mathrm{Ind}_D^{BC} V)_n \qquad \text{for all } n \geq m+1$$

by Theorem 2.10, and so the character of V_n is given by the restriction of F_V to D_n in this range. The theorem follows. □

Corollary 4.17. (Polynomial growth of dimension for finitely generated $\mathrm{FI}_{\mathcal{W}}$–modules). *Given a finitely generated $FI_{\mathcal{W}}$–module V over a field of characteristic zero with associated character polynomial F_V, the dimension dim(V_n) of V_n is given by $F_V(n,0,0,0,\ldots)$ in the stable range. In particular, if V is finitely generated in degree $\leq d$, then dim(V_n) is eventually a polynomial in n of degree at most d.*

Corollary 4.18. (Characters only depend on short cycles). *Suppose that k is a field of characteristic zero, and let V be a finitely generated $FI_{\mathcal{W}}$–module. Let χ_n denote the character of the B_n–representation V_n. Then there exists some positive integer $d \leq$ weight(V), independent of n, such that for every $w \in \mathcal{W}_n$, the value $\chi_n(w)$ depends only on cycles in w of length at most d.*

Remark 4.19. (Character polynomials of co–$\mathrm{FI}_{\mathcal{W}}$–modules). Suppose that V is a co–$\mathrm{FI}_{\mathcal{W}}$–module over a field of characteristic 0. We define its dual V^* to be the $\mathrm{FI}_{\mathcal{W}}$–module with $(V^*)_n = (V_n)^*$. Suppose V^* is a finitely generated $\mathrm{FI}_{\mathcal{W}}$–module of weight $\leq d$ and stability degree $\leq s$, and that F_V is the associated character polynomial. Since $(V_n)^* \cong (V_n)$ (see Geck–Pfeiffer [GP00, Corollary 3.2.14]), the characters of $\chi_{V_n} = F_V$ in the range $n \geq s+d$.

4.4 Polynomial dimension over positive characteristic

Church–Ellenberg–Farb–Nagpal proved that the dimensions of finitely generated FI_A–modules over a field k are eventually polynomial even when k has positive characteristic [CEFN14, Theorem 1.2]. We use their result to prove the same for all $\mathrm{FI}_{\mathcal{W}}$–modules.

Theorem 4.20. (Polynomial growth of dimension over arbitrary fields). *Let k be any field, and let V be a finitely generated $\mathrm{FI}_{\mathcal{W}}$–module over k. Then there exists an integer-valued polynomial $P(T) \in \mathbb{Q}[T]$ such that*

$$\mathit{dim}_k(V_n) = P(n) \qquad \textit{for all } n \textit{ sufficiently large.}$$

We note that, in contrast to the result over characteristic zero, Theorem 4.20 does not come with bounds on the degree of $P(T)$ or the range of n-values for which the equality holds.

Proof of Theorem 4.20. When V is a finitely generated FI_A–module, the result follows from [CEFN14, Theorem 1.2]. If V is a finitely generated FI_{BC} or FI_D–module, then by Proposition 2.8 its restriction to FI_A is finitely generated, and the result again follows from [CEFN14, Theorem 1.2]. □

4.5 The character polynomials of $\mathrm{FI}_{\mathcal{W}}\sharp$–modules

In this section we compute the character polynomials of the FI_{BC}–modules $M_{BC}(U)$, Proposition 4.21. We conclude that the character polynomial of an $\mathrm{FI}_{BC}\sharp$–module V must equal χ_{V_n} for *all* values of n. The formula given in Proposition 4.21 is moreover useful for computing character polynomials of $\mathrm{FI}_{BC}\sharp$–modules, such as in our applications in Sections 5.1 and 5.2. We end this section with Proposition 4.23, the character polynomials of the $\mathrm{FI}_{\mathcal{W}}$–modules $M_{\mathcal{W}}(\mathbf{m})$ for each family $\mathcal{W}_n$.

Proposition 4.21. (The Character of $M_{BC}(U)_n$). *Let k be a field of characteristic zero. Let U be a representation of B_m with character χ^U. Then the character $\chi^{M_{BC}(U)_n}$ is, for each n, given by the character polynomial P^U:*

$$P^U(w) = \left(\sum_{\substack{(\alpha,\beta) \\ |\alpha|+|\beta|=m}} \chi^U_{(\alpha,\beta)} \binom{\mathbf{X}}{\alpha}\binom{\mathbf{Y}}{\beta} \right)(w)$$

$$:= \sum_{\substack{(\alpha,\beta) \\ |\alpha|+|\beta|=m}} \chi^U_{(\alpha,\beta)} \binom{X_1(w)}{n_1(\alpha)}\binom{X_2(w)}{n_2(\alpha)} \cdots \binom{X_m(w)}{n_m(\alpha)}\binom{Y_1(w)}{n_1(\beta)}\binom{Y_2(w)}{n_2(\beta)} \cdots \binom{Y_m(w)}{n_m(\beta)}$$

Proof of Proposition 4.21. Since $M_{BC}(U)_n = \mathrm{Ind}_{B_m \times B_{n-m}}^{B_n} U \boxtimes k$, with k the trivial B_{n-m}–representation, the result follows from Lemma 4.15. $\square$

Corollary 4.22. *Let V be an $FI_{BC}\sharp$–module V over a field of characteristic zero. Then if V is finitely generated in degree $\leq d$, the characters of V_n are equal to a unique character polynomial $F_V \in k[X_1, Y_1, X_2, Y_2, \ldots]$ of degree at most d, with equality for every value of $n \geq 0$. The dimensions of V_n are given by a polynomial of degree at most d*

$$\mathit{dim}_k(V_n) = F_V(n, 0, 0, \ldots) \qquad \textit{for every value of } n.$$

We can find explicit formulas for the $\mathrm{FI}_{\mathcal{W}}$–modules $M_{\mathcal{W}}(\mathbf{m})$.

Proposition 4.23. *Let k be $\mathbb{Z}$ or a field of characteristic zero. When $\mathcal{W}_n$ is S_n or B_n, the character polynomials of $M_{\mathcal{W}}(\mathbf{m})$ are:*

$$\chi^{M_A(\mathbf{m})} = m!\binom{X_1}{m} \qquad\qquad \chi^{M_{BC}(\mathbf{m})} = 2^m m!\binom{X_1}{m}.$$

When $\mathcal{W}_n$ is D_n, $M_D(\mathbf{m})$ is also given by a character polynomial for $n > m$:

$$\chi^{M_D(\mathbf{m})} = 2^m m!\binom{X_1}{m} \qquad \textit{when } n > m.$$

When $n = m$, the character of $M_D(\mathbf{m})_m$ takes the value $2^{m-1}m!$ on the identity and vanishes otherwise.

Proof of Proposition 4.23. Take as basis for $M_{\mathcal{W}}(\mathbf{m})_n$ the set $S = \{e_f \mid f \in \mathrm{Hom}_{\mathrm{FI}_{\mathcal{W}}}(\mathbf{m}, \mathbf{n})\}$.

An element $w \in \mathcal{W}_n$ will permute these basis elements; the trace of w is the size of its fixed set in S. A basis element e_f is fixed by w only if w fixes its image $f(\mathbf{m}) \subseteq \mathbf{n}$ pointwise; conversely for every choice of m (positive) 1-cycles

$$(a_1)(-a_1), (a_2)(-a_2), \ldots, (a_m)(-a_m)$$

in w, w will fix all basis elements e_f for which the image of f is

$$f(\mathbf{m}) = \{\pm a_1, \ldots, \pm a_m\} \subseteq \mathbf{n}.$$

When $\mathcal{W}_n$ is S_n, there are $m!$ such maps. When $\mathcal{W}_n$ is B_n, there are $2^m m!$ such maps. When $\mathcal{W}_n$ is D_n, there are $2^m m!$ such maps whenever $n > m$; when $n = m$ there are only $2^{m-1}m!$, since in this case each endomorphism f must reverse an even number of signs. The formulas follow. $\square$

Example 4.24. Using the formulas for the characters of $M_A(\mathbf{m})$ and $M_{BC}(\mathbf{m})$ in Proposition 4.23 and a standard binomial–multinomial identity, one can decompose the characters of $M_A(\mathbf{m}) \otimes M_A(\mathbf{p})$ and $M_{BC}(\mathbf{m}) \otimes M_{BC}(\mathbf{p})$. By Corollary 3.12, these characters completely determine the $\mathrm{FI}_{\mathcal{W}}$–structure:

$$M_A(\mathbf{m}) \otimes M_A(\mathbf{p}) = \bigoplus_{d=0}^{m} \frac{m!\, p!}{(m+p-d)!} \binom{m+p-d}{d, m-d, p-d} M_A(\mathbf{m}+\mathbf{p}-\mathbf{d})$$

$$M_{BC}(\mathbf{m}) \otimes M_{BC}(\mathbf{p}) = \bigoplus_{d=0}^{m} \frac{2^d\, m!\, p!}{(m+p-d)!} \binom{m+p-d}{d, m-d, p-d} M_{BC}(\mathbf{m}+\mathbf{p}-\mathbf{d})$$

5 Some applications

$\mathrm{FI}_{\mathcal{W}}$–modules arise naturally in numerous areas of mathematics. In this section we give some applications of the theory developed in this paper to the cohomology of the group of pure string motions and the cohomology of the complements of the Weyl groups' reflecting hyperplanes.

5.1 The cohomology of the group of pure string motions

In [Wil12], we proved that the cohomology of the pure string motion groups $P\Sigma_n$ is uniformly representation stable with respect to a natural action of the hyperoctahedral group.

The group Σ_n of string motions is a generalization of the braid group. It is defined as the group of *motions* of n smoothly embedded, oriented, unlinked, unknotted circles in $\mathbb{R}^3$; see for example Brownstein–Lee [BL93] for a complete definition.

Hatcher–Wahl [HW10] proved that the string motion groups are (integrally) homologically stable, as a particular instance of their homologically stability results for mapping class groups of 3–manifolds. Brendle–Hatcher [BH11] realized the string motion group as the fundamental groups of certain configuration spaces of rigid circles. These configuration spaces are not $K(\pi, 1)$'s, and Kupers proved that the spaces are themselves homological stable [Kup13].

The work of Dahm (see Dahm [Dah62] or Goldsmith [Gol81]) identifies Σ_n with the *symmetric automorphism group* of the free group F_n on n letters $x_1, \ldots, x_n$, the subgroup of automorphisms generated by the following elements:

$$\alpha_{i,j} = \begin{cases} x_i \mapsto x_j x_i x_j^{-1} \\ x_\ell \mapsto x_\ell \quad (\ell \neq i) \end{cases} \qquad \tau_i = \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_i \\ x_\ell \mapsto x_\ell \quad (\ell \neq i,\, i+1) \end{cases} \qquad \rho_i = \begin{cases} x_i \mapsto x_i^{-1} \\ x_\ell \mapsto x_\ell \quad (\ell \neq i) \end{cases}$$

The subgroup $P\Sigma_n = \langle \alpha_{i,j} \rangle \subseteq \Sigma_n$ is the group of *pure* symmetric automorphisms (or *pure* string motions), the analogue of the pure braid group.

The central theorem of [Wil12]:

Theorem 5.1. *[Wil12, Theorem 6.1] For each fixed $m \geq 0$, the sequence of B_n–representations $\{H^m(P\Sigma_n;\mathbb{Q})\}_n$ is uniformly representation stable with respect to the maps*

$$\phi_n \,:\, H^m(P\Sigma_n;\mathbb{Q}) \to H^m(P\Sigma_{n+1};\mathbb{Q})$$

induced by the 'forgetful' map $P\Sigma_{n+1} \to P\Sigma_n$. The sequence stabilizes once $n \geq 4m$.

The theory of FI_{BC}–modules developed here allows for a significantly simplified proof of this result, and new perspective on the structure of these cohomology groups.

The integral homology group $H^1(P\Sigma_n;\mathbb{Z}) = P\Sigma_n/[P\Sigma_n, P\Sigma_n]$ is the free abelian group $\mathbb{Z}[\ \alpha_{i,j} \mid i \neq j\ ]$, and the cohomology ring is generated by the dual elements $\alpha^*_{i,j}$. A presentation for the integral cohomology was conjectured by Brownstein and Lee [BL93, Conjecture 4.6] and proven by Jensen, McCammond, and Meier [JMM06, Theorem 6.7] (see also Griffin [Gri13b, Section 4]). Jensen–McCammond–Meier study the action of $P\Sigma_n/\text{Inn}(F_n)$ on the MacCullough–Miller complex [MM96] to obtain Theorem 5.2.

Theorem 5.2. *[JMM06, Theorem 6.7]. The cohomology ring $H^*(P\Sigma_n;\mathbb{Z})$ is the exterior algebra generated by the degree-one classes $\alpha^*_{i,j}$, with $i, j \in [n]$, $i \neq j$, modulo the relations*

$$(1)\ \alpha^*_{i,j} \wedge \alpha^*_{j,i} = 0 \qquad\qquad (2)\ \alpha^*_{\ell,j} \wedge \alpha^*_{j,i} - \alpha^*_{\ell,j} \wedge \alpha^*_{\ell,i} + \alpha^*_{i,j} \wedge \alpha^*_{\ell,i} = 0$$

In [Wil12], to prove that the sequence $H^m(P\Sigma_n;\mathbb{Q})$ is uniformly representation stable, we use a combinatorial description of the cohomology groups given by Jensen–McCammond–Meier [JMM06] and an orbit–stabilizer argument to decompose each group into a sum of induced representation of a particular form. We then use a result of Church–Farb [CF13, Theorem 4.6] (inspired by the work of Hemmer [Hem10, Theorem 2.4]), to deduce from the combinatorics of the branching rules that these induced representations are uniformly representation stable.

Here, we can recover uniform representation stability for $H^m(P\Sigma_\bullet;\mathbb{Q})$ as a B_n–representation almost immediately by demonstrating that it is finitely generated as an $\text{FI}_{BC}\sharp$–module, as follows.

Theorem 5.3. *Let k be $\mathbb{Z}$ or $\mathbb{Q}$. The cohomology rings $H^*(P\Sigma_\bullet, k)$ form an $FI_{BC}\sharp$–module, and a graded FI_{BC}–algebra of finite type, with $H^m(P\Sigma_\bullet, k)$ finitely generated in degree $\leq 2m$. In particular the FI_{BC}–algebra $H^*(P\Sigma_\bullet,\mathbb{Q})$ has slope ≤ 2.*

Proof of Theorem 5.3. The map induced by an $\text{FI}_{BC}\sharp$ morphism $f : \mathbf{m_0} \to \mathbf{n_0}$ on $H^1(P\Sigma_\bullet, \mathbb{Z})$ is:

$$\begin{aligned} f_* : \; H^1(P\Sigma_m; k) &\longrightarrow \; H^1(P\Sigma_n; k) \\ \alpha^*_{i,j} &\longmapsto \begin{cases} \;\;\alpha^*_{|f(i)|,|f(j)|}, & \text{if } f(i) \neq 0, f(j) > 0 \\ -\alpha^*_{|f(i)|,|f(j)|}, & \text{if } f(i) \neq 0, f(j) < 0 \\ \;\;0, & \text{if } f(i) = 0 \text{ or } f(j) = 0 \end{cases} \end{aligned}$$

It is straightforward to verify that f_* extends to an algebra map on $H^*(P\Sigma_\bullet, k)$, and that this action is functorial.

The FI_{BC}–module $H^1(P\Sigma_\bullet; k)$ is generated in degree 2 by $\alpha_{1,2}$, and $V^*_\bullet = H^*(P\Sigma_\bullet; k)$ is generated as an FI_{BC}–algebra by $H^1(P\Sigma_\bullet; k)$. We conclude from Proposition 2.23 that $V^m_\bullet$ is finitely generated in degree $\leq 2m$, and that $H^*(P\Sigma_\bullet; \mathbb{Q})$ is a graded FI_{BC}–algebra of slope ≤ 2. ☐

Propositions 2.8(1) and (2) imply that $V^*_\bullet$ restricts to a graded FI_A and FI_D–algebra of finite type, with $V^m_\bullet$ generated in degree $\leq 2m$.

Since $V^m_\bullet = H^m(P\Sigma_\bullet; k)$ is a $\text{FI}_{BC}\sharp$–module generated in degree $\leq 2m$, by Corollary 3.11 the sequence is uniformly representation stable in degree $4m$, as B_n–representations or as S_n–representations.

Corollary 5.4. *For each m, the sequence $\{H^m(P\Sigma_n; \mathbb{Q})\}_n$ of representations of B_n (or S_n) is uniformly representation stable, stabilizing once $n \geq 4m$.*

Remark 5.5. We only defined *representation stability* for $\text{FI}_\mathcal{W}$–modules over fields of characteristic zero (Definition 2.1). However, the presentation for the groups $H^m(P\Sigma_n; \mathbb{Z})$ given by Jensen-McCammond–Meier shows that these groups are free abelian ([JMM06, Theorem 6.7], see Theorem 5.2), and we can identify $H^m(P\Sigma_n; \mathbb{Z})$ with the integer span of the basis $\alpha^*_{i_1,j_1} \wedge \cdots \wedge \alpha^*_{i_m,j_m}$ for $H^m(P\Sigma_n; \mathbb{Q})$. Hence we get a version of representation stability for the integral groups $H^m(P\Sigma_n; \mathbb{Z})$ by redefining the subrepresentation $V(\lambda)_n$ as the integral Specht module associated to $\lambda[n]$.

Theorem 4.16 implies that the characters of the sequence $\{H^m(P\Sigma_n; \mathbb{Q})\}_n$ are given by a character polynomial of degree $\leq 2m$. As in the above remark, since the integral cohomology is free abelian, these same character polynomials give characters for the integral cohomology.

Corollary 5.6. *Let k be $\mathbb{Z}$ or $\mathbb{Q}$. Fix an integer $m \geq 0$. The characters of the sequence of B_n–representations $\{H^m(P\Sigma_n; k)\}_n$ are given, for all values of n, by a unique character polynomial of degree $\leq 2m$.*

This concludes a simpler proof of [Wil12, Theorems 6.1 and 6.4]. We have moreover extended the results of [Wil12] to integer coefficients, and obtained polynomiality results on the characters.

For small values of m, we can compute the $\mathrm{FI}_{BC}\sharp$ and $\mathrm{FI}_A\sharp$–module structures and character polynomials of $H^m(P\Sigma_\bullet;\mathbb{Z})$ by computing traces on an explicit basis for $H^m(P\Sigma_n;\mathbb{Z})$, $n = 1,\ldots,2m$, and using Proposition 4.21. The result is, as an $\mathrm{FI}_{BC}\sharp$–module,

$$H^1(P\Sigma_\bullet;\mathbb{Z}) = M_{BC}\left(\ydiagram{1}\,,\ydiagram{1}\right) \qquad \chi_{H^1(P\Sigma_\bullet;\mathbb{Z})} = 2\binom{X_1}{2} - 2\binom{Y_1}{2}$$

In degree 2:

$$\begin{aligned} H^2(P\Sigma_\bullet;\mathbb{Z}) = {} & M_{BC}\left(\ydiagram{1,1}\,,\varnothing\right) \oplus M_{BC}\left(\ydiagram{2,1}\,,\varnothing\right) \oplus M_{BC}\left(\ydiagram{1}\,,\ydiagram{1,1}\right) \oplus M_{BC}\left(\ydiagram{1}\,,\ydiagram{2}\right) \\ & \oplus M_{BC}\left(\ydiagram{1,1}\,,\ydiagram{2}\right) \oplus M_{BC}\left(\ydiagram{2}\,,\ydiagram{1,1}\right) \end{aligned}$$

$$\begin{aligned} \chi_{H^2(P\Sigma_\bullet;\mathbb{Z})} = {} & 12\binom{X_1}{4} + 12\binom{Y_1}{4} + 9\binom{X_1}{3} + 9\binom{Y_1}{3} - 4\binom{X_2}{2} + 4\binom{Y_2}{2} \\ & - 4\binom{X_1}{2}\binom{Y_1}{2} - X_1X_2 - X_1Y_2 - X_2Y_1 - Y_1Y_2 - \binom{X_1}{2}Y_1 - X_1\binom{Y_1}{2} \end{aligned}$$

By restricting to the action of the symmetric groups we find, as an $\mathrm{FI}_A\sharp$–module,

$$H^1(P\Sigma_\bullet;\mathbb{Z}) = M_A\left(\ydiagram{2}\right) \oplus M_A\left(\ydiagram{1,1}\right) \qquad \chi_{H^1(P\Sigma_\bullet;\mathbb{Z})} = 2\binom{X_1}{2}$$

$$H^2(P\Sigma_\bullet;\mathbb{Z}) = M_A\left(\ydiagram{1,1,1}\right)^{\bigoplus 2} \oplus M_A\left(\ydiagram{2,1}\right)^{\bigoplus 3} \oplus M_A\left(\ydiagram{3}\right) \oplus M_A\left(\ydiagram{2,1,1}\right)^{\bigoplus 2} \oplus M_A\left(\ydiagram{3,1}\right)^{\bigoplus 2}$$

$$\chi_{H^2(P\Sigma_\bullet;\mathbb{Z})} = 12\binom{X_1}{4} + 9\binom{X_1}{3} - X_1X_2 - 4\binom{X_2}{2}$$

Problem 5.7. For each m, compute the B_n character polynomial of $H^m(P\Sigma_\bullet,\mathbb{Z})$, and compute its decomposition as an $\mathrm{FI}_{BC}\sharp$–module into a sum of induced representations $M_{BC}(U)$.

5.1.1 Generalizations

There are several families of groups that naturally generalize the (pure) braid groups and (pure) symmetric automorphism groups, which we outline below. With each family, there are open questions concerning whether the cohomology rings admit the structure of a finite type $\text{FI}_{\mathcal{W}}$ or $\text{FI}_{\mathcal{W}}\sharp$–algebra, and how this structure reflects the structure of the groups.

Partially symmetric automorphisms. The group Σ_n^k of *partially symmetric automorphisms* of the free group $F_n = \langle x_1, \ldots, x_n \rangle$ are those automorphisms that send each of the first k generators $x_1, \ldots, x_k$ to a conjugate of one of the elements x_1, x_1^{-1}, $\ldots, x_k, x_k^{-1}$. We impose no restrictions on the images of $x_{k+1}, \ldots, x_n$. The *pure partially symmetric automorphism group* $P\Sigma_n^k$ is the subgroup of Σ_n^k of automorphisms that send each generator x_j with $1 \leq j \leq k$ to a conjugate of itself. We note that

$$\Sigma_n^n = \Sigma_n, \qquad P\Sigma_n^n = P\Sigma_n, \quad \text{and} \quad P\Sigma_n^0 = \Sigma_n^0 = \text{Aut}(F_n);$$

these groups interpolate between the (pure) symmetric automorphism group and the full automorphism group of F_n.

The groups $P\Sigma_n^k$ were studied by Jensen–Wahl [JW04] for their relationships to mapping class groups. Jensen–Wahl have computed a presentation and established certain homological properties of the groups. Bux–Charney–Vogtmann [BCV09] determined that the image of the group $P\Sigma_n^k$ in $\text{Out}(F_n)$ has virtual cohomological dimension $2n - k - 2$ when $k \neq 0$. They exhibit a proper action of these outer automorphism groups on a $(2n - k - 2)$–dimensional deformation retract of a certain contractible subcomplex of the spine of Culler–Vogtmann's Outer space; see Charney–Vogtmann [CV09] for details.

Zaremsky [Zar14] proved that both families $P\Sigma_n^k$ and Σ_n^k are, for fixed k, rationally homologically stable in n. He proved moreover that for fixed n, the groups Σ_{n+k}^k are rationally homologically stable in k. Zaremsky obtains these results by studying the groups' actions on subcomplexes of the spine of Auter space. He uses methods from discrete Morse theory to prove that the filtered pieces of certain subcomplexes are highly connected, extending techniques of McEwen–Zaremsky [MZ12].

Given these results, it would be interesting to determine whether there is a FI_{BC} or $\text{FI}_{BC}\sharp$–module structure on the rational cohomology groups of $P\Sigma_{n+k}^k$ as a sequence in k, and, if so, to determine the associated stable decompositions and character polynomials.

Symmetric automorphisms of free products. Given a group G, let G^{*n} denote its n-fold free product

$$G^{*n} := \underbrace{G * G * \cdots * G}_{n \text{ copies}}.$$

The automorphism group $\text{Aut}(G^{*n})$ contains a copy of the symmetric group S_n

which permutes the n free factors. These permutations normalize the following subgroups of automorphisms; see for example Griffin [Gri13a, Gri13b] for details.

- The *Fouxe-Rabinovitch group* $\mathrm{FR}(G^{*n}) \subseteq \mathrm{Aut}(G^{*n})$ generated by *partial conjugations* of G^{*n}. A partial conjugation is an automorphism that conjugates the i^{th} free factor G by some g in the j^{th} factor G with $i \neq j$. All factors other than the i^{th} are fixed.
- The inner automorphisms of each factor $\prod_n \mathrm{Inn}(G)$
- All automorphisms of each factor $\prod_n \mathrm{Aut}(G)$
- The *Whitehead automorphism group* $\mathrm{Wh}(G^{*n}) := \mathrm{FR}(G^{*n}) \rtimes \prod_n \mathrm{Inn}(G)$
- The *pure automorphism group* $\mathrm{PAut}(G^{*n}) := \mathrm{FR}(G^{*n}) \rtimes \prod_n \mathrm{Aut}(G)$

The *symmetric automorphism group* of G^{*n} is the group

$$\Sigma\mathrm{Aut}(G^{*n}) := \big(\mathrm{PAut}(G^{*n}) \rtimes S_n\big).$$

We note that

$$P\Sigma_n \cong \mathrm{FR}(\mathbb{Z}^{*n}) \cong \mathrm{Wh}(\mathbb{Z}^{*n}) \qquad \text{and} \qquad \Sigma_n \cong \Sigma\mathrm{Aut}(\mathbb{Z}^{*n}).$$

Griffin constructs a classifying space for $\mathrm{FR}(G^{*n})$, which he defines as a moduli space of *cactus products*, and alternatively characterizes combinatorially in terms of *diagonal complexes* comprised of *forest posets*. Using this classifying space he computes the homology of the groups $\mathrm{FR}(G^{*n})$, $\mathrm{PAut}(G^{*n})$, and $\Sigma\mathrm{Aut}(G^{*n})$.

Collinet–Djament–Griffin [CDG12] have proven that if G does not contain $\mathbb{Z}$ as a free factor, the sequences $\mathrm{Aut}(G^{*n})$ and $\Sigma\mathrm{Aut}(G^{*n})$ are (integrally) homologically stable, stabilizing in degree i once $n \geq 2i+2$. Their work complements the results of Hatcher [Hat95] for $G \cong \mathbb{Z}$ and extends results of Hatcher–Wahl [HW10] for several important classes of groups G coming from low-dimensional topology. Collinet–Djament–Griffin prove their results using the theory of functor homology, and an analysis of the action of $\mathrm{FR}(G^{*n})$ on a variation of the MacCullough–Miller complex [MM96] due to Chen–Glover–Jensen [CGJ05].

We would be interested to better understand the relationship between the work done on the groups $\mathrm{FR}(G^{*n})$, $\mathrm{Wh}(G^{*n})$, and $\mathrm{PAut}(G^{*n})$ and the theory of FI_A–modules.

Virtual and flat braid groups. The *(pure) virtual braid group* and the *(pure) flat braid group* are generalizations of the (pure) braid group that allow *virtual* or *flat* crossings of strands. This additional structure was introduced by Kauffman [Kau99], motivated by the study of knots in thickened higher-genus surfaces and the combinatorial theory of Gauss codes. Virtual and flat crossings are distinct from the under– and over–crossings in familiar knot and braid diagrams, and each have their own admissible Redemeister moves. For details see for example Kauffman [Kau99, Kau00], Vershinin [Ver01], Kauffman–Lambropoulou [KL04], Bardakov [Bar04], and Bartholdi–Enriquez–Etingof–Rains [BEER06].

In [Lee13], Peter Lee analyzes the cohomology of the pure virtual braid groups and the pure flat braid groups as representations of the symmetric groups. He proves that, for both families, the rational cohomology groups are uniformly representation stable [Lee13, Corollaries 1 and 5], and his work suggests that these cohomology sequences are in fact $\mathrm{FI}_A\sharp$–algebras. His results raise questions about the structure of these $\mathrm{FI}_A\sharp$–algebras and their associated character polynomials, and the structure of the integral cohomology groups.

5.2 The cohomology of hyperplane complements

Let $\mathcal{W}_n$ be the Weyl group in type A_{n-1}, $\mathrm{B}_n/\mathrm{C}_n$, or D_n, and consider the canonical action of $\mathcal{W}_n$ on $\mathbb{C}^n$ by (signed) permutation matrices. Let $\mathscr{A}(n)$ be the set of hyperplanes fixed by the (complexified) reflections of $\mathcal{W}_n$, and let $\mathscr{M}_{\mathcal{W}} = \mathscr{M}_{\mathcal{W}}(n)$ be their complement

$$\mathscr{M}_{\mathcal{W}}(n) := \mathbb{C}^n \setminus \bigcup_{H \in \mathscr{A}(n)} H.$$

The group $\mathcal{W}_n$ permutes the set of hyperplanes, and acts on $\mathscr{M}_{\mathcal{W}}$. For each family $\{\mathcal{W}_n\}$, the hyperplane complements can be described explicitly:

$$\begin{aligned}
\mathscr{M}_A(n) &= \{(v_1, \ldots, v_n) \in \mathbb{C}^n \mid v_i \neq v_j \text{ for } i \neq j\} \\
\mathscr{M}_D(n) &= \{(v_1, \ldots, v_n) \in \mathbb{C}^n \mid v_i \neq \pm v_j \text{ for } i \neq j\} \\
\mathscr{M}_{BC}(n) &= \{(v_1, \ldots, v_n) \in \mathbb{C}^n \mid v_i \neq \pm v_j \text{ for } i \neq j; v_i \neq 0 \text{ for all } i\}
\end{aligned}$$

We note that $\mathscr{M}_{BC}(n) \subseteq \mathscr{M}_D(n) \subseteq \mathscr{M}_A(n)$.

The hyperplane complement $\mathscr{M}_A(n)$ is the ordered n–point configuration space of the plane $\mathbb{C}$; it is an Eilenberg–Mac Lane space with fundamental group the pure braid group on n strands. Arnol'd computed its integral cohomology in 1969 [Arn69]. Its quotient $\mathscr{M}_A(n)/S_n$ is an Eilenberg–Mac Lane space with fundamental group the braid group on n strands. Brieskorn showed that $\mathscr{M}_{BC}(n)$ and $\mathscr{M}_D(n)$ and their quotients $\mathscr{M}_{BC}(n)/B_n$ and $\mathscr{M}_D(n)/S_n$ are also Eilenberg–Mac Lane spaces

[Bri73, Proposition 2]; their fundamental groups are sometimes called *generalized (pure) braid groups*.

Brieskorn [Bri73] and Orlik–Solomon [OS80] studied the cohomology of the complement $\mathscr{M}$ of a general arrangement of complex hyperplanes containing the origin. Define a set of hyperplanes $H_1, \ldots, H_p$ to be *dependent* if $\operatorname{codim}(H_1 \cap \cdots \cap H_p) < p$.

Let $E(\mathscr{A})$ to be the complex exterior algebra $E(\mathscr{A}) := \bigwedge \langle e_H \mid H \in \mathscr{A} \rangle$ and let $I(\mathscr{A}) \subseteq E(\mathscr{A})$ be the ideal

$$I(\mathscr{A}) := \langle \quad \sum_{\ell=1}^{p} (-1)^\ell \, e_{H_1} \cdots \widehat{e_{H_\ell}} \cdots e_{H_p} \quad | \quad H_1, \ldots, H_p \text{ dependent} \quad \rangle$$

Orlik–Solomon proved that $H^*(\mathscr{M}_\mathcal{W}, \mathbb{C})$ is isomorphic to $A(\mathscr{A}) := E(\mathscr{A})/I(\mathscr{A})$ as a graded algebra [OS80, Theorem 5.2]. Their work implies that $H^*(\mathscr{M}_\mathcal{W}, \mathbb{C}) \cong A(\mathscr{A})$ as a graded $\mathbb{C}[\mathcal{W}_n]$–module under the $\mathcal{W}_n$–action $w \cdot e_H = e_{wH}$.

The structure of $H^*(\mathscr{M}_A(n), \mathbb{C})$ as an S_n–representation is described by Lehrer–Solomon [LS86], and the structure of the B_n–representations $H^*(\mathscr{M}_{BC}(n), \mathbb{C})$ is described by Douglass [Dou92]. Lehrer–Solomon and Douglass give decompositions of the $\mathcal{W}_n$–representations $H^*(\mathscr{M}_\mathcal{W}(n), \mathbb{C})$ in type A and B/C, respectively, as sums of certain explicitly described induced representations. Lehrer–Solomon conjectured that, as they prove in type A, the cohomology groups $H^m(\mathscr{M}_\mathcal{W}, \mathbb{C})$ decompose into a sum of induced one-dimensional representations of centralizers, summed over the set of $\mathcal{W}_n$ conjugacy classes [LS86, Conjecture 1.6]. Recent progress has been made on this conjecture; see Douglass–Pfeiffer–Röhrle [DPR12].

Church and Farb prove that, for each degree m, the sequence $H^m(\mathscr{M}_A(n), \mathbb{Q})$ is a uniformly representation stable sequence of S_n–representations [CF13, Theorem 4.1]. Church–Ellenberg–Farb further prove that $H^m(\mathscr{M}_A, \mathbb{Q})$ is a graded FI♯–algebra of finite type; this is a special case of their much more general results on the ordered configuration space of manifolds [CEF12, Theorem 4.7; see also Theorems 4.1 and 4.2]. In [CF13, Theorem 4.6], Church–Farb analyze the stability behaviour of the sequence $H^m(\mathscr{M}_{BC}, \mathbb{C})$ of B_n–representations.

The following result recovers [CF13, Theorem 4.1 and 4.6] in types A_{n-1} and $\mathrm{B}_n/\mathrm{C}_n$. It extends the work of Church–Ellenberg–Farb on the cohomology of the ordered configuration space of $\mathbb{C}$.

Theorem 5.8. *Let $\mathscr{M}_\mathcal{W}$ be the complex hyperplane complement associated with the Weyl group $\mathcal{W}_n$ in type A_{n-1}, B_n/C_n, or D_n, as described above. In each degree m, the cohomology groups $H^m(\mathscr{M}_A(\bullet), \mathbb{C})$ form an FI_A♯–module finitely generated in degree $\leq 2m$, and both $H^m(\mathscr{M}_{BC}(\bullet), \mathbb{C})$ and $H^m(\mathscr{M}_D(\bullet), \mathbb{C})$ are FI_{BC}♯–modules finitely generated in degree $\leq 2m$. For each $\mathcal{W}$, the cohomology $H^*(\mathscr{M}_\mathcal{W}(\bullet), \mathbb{C})$ of the hyperplane complements is a graded $FI_\mathcal{W}$–module of slope 2.*

Proof of Theorem 5.8. For each $\mathcal{W}$, the projection map P has a section S:

$$P : \mathscr{M}_{\mathcal{W}}(n+1) \longrightarrow \mathscr{M}_{\mathcal{W}}(n) \qquad S : \mathscr{M}_{\mathcal{W}}(n) \longrightarrow \mathscr{M}_{\mathcal{W}}(n+1)$$
$$(v_1, \ldots, v_n, v_{n+1}) \longmapsto (v_1, \ldots, v_n) \qquad (v_1, \ldots, v_n) \longmapsto (v_1, \ldots, v_n, 1 + \sum_{i=1}^{n} |v_i|)$$

and so P induces an injective map on cohomology, as follows. We associate each hyperplane $H \subseteq \mathbb{C}^n$ to its orthogonal complement, the span of the vectors

$$\pm(\mathbf{e}_i - \mathbf{e}_j), \quad \pm(\mathbf{e}_i + \mathbf{e}_j), \quad \text{or} \quad \pm\, \mathbf{e}_i \qquad \text{for } i, j = 1, \ldots, n.$$

The inclusion of these normal vectors $\mathbb{C}^n \to \mathbb{C}^{n+1}$ gives an identification of the hyperplane $H \subseteq \mathbb{C}^n$ with a hyperplane $H \subseteq \mathbb{C}^{n+1}$, which define the induced map P^*.

$$P^* : H^*(\mathscr{M}_{\mathcal{W}}(n); \mathbb{C}) \longrightarrow H^*(\mathscr{M}_{\mathcal{W}}(n+1), \mathbb{C})$$
$$e_H \longmapsto e_H$$

These inclusions are $\mathcal{W}_n$–equivariant maps, and give $H^*(\mathscr{M}_{\mathcal{W}}(\bullet); \mathbb{C})$ the structure of a graded $\mathrm{FI}_{\mathcal{W}}$–module.

The FI_A–module $H^1(\mathscr{M}_A(\bullet); \mathbb{C})$ is finitely generated in degree ≤ 2 by element $e_{(\mathbf{e}_1 - \mathbf{e}_2)^\perp}$, and the FI_{BC}–module $H^1(\mathscr{M}_{BC}(\bullet); \mathbb{C})$ is finitely generated in degree ≤ 2 by elements $e_{(\mathbf{e}_1 - \mathbf{e}_2)^\perp}$, $e_{(\mathbf{e}_1 + \mathbf{e}_2)^\perp}$, and $e_{(\mathbf{e}_1)^\perp}$. It follows from Proposition 2.23 that $H^m(\mathscr{M}_{\mathcal{W}}(\bullet); \mathbb{C})$ is finitely generated in degree $\leq 2m$ in types A and B/C. The bound on the slope of the $\mathrm{FI}_{\mathcal{W}}$–algebra $H^*(\mathscr{M}_{\mathcal{W}}(\bullet); \mathbb{C})$ follows from Theorem 2.12.

The section S induces a map $S^* : H^*(\mathscr{M}_{\mathcal{W}}(n+1); \mathbb{C}) \longrightarrow H^*(\mathscr{M}_{\mathcal{W}}(n), \mathbb{C})$; when $\mathcal{W}_n$ is S_n or B_n these sections give $H^*(\mathscr{M}_{\mathcal{W}}(\bullet), \mathbb{C})$ the structure of an $\mathrm{FI}_{\mathcal{W}}\sharp$–module, just as in the proof of [CEF12, Theorem 4.6]. We can describe this structure explicitly: an $\mathrm{FI}_{BC}\sharp$–morphism $f : \mathbf{m_0} \to \mathbf{n_0}$ acts on the generators e_H as follows.

$$e_{(\mathbf{e}_i - \mathbf{e}_j)^\perp} \longmapsto \begin{cases} e_{(\mathbf{e}_{f(i)} - \mathbf{e}_{f(j)})^\perp}, & \text{if } f(i), f(j) \neq 0 \\ 0, & \text{if } f(i) = 0 \text{ or } f(j) = 0 \end{cases}$$

$$e_{(\mathbf{e}_i + \mathbf{e}_j)^\perp} \longmapsto \begin{cases} e_{(\mathbf{e}_{f(i)} + \mathbf{e}_{f(j)})^\perp}, & \text{if } f(i), f(j) \neq 0 \\ 0, & \text{if } f(i) = 0 \text{ or } f(j) = 0 \end{cases}$$

$$e_{(\mathbf{e}_i)^\perp} \longmapsto \begin{cases} e_{(\mathbf{e}_{f(i)})^\perp}, & \text{if } f(i) \neq 0 \\ 0, & \text{if } f(i) = 0 \end{cases}$$

Here, we use the convention that $\mathbf{e}_{-i} := -\mathbf{e}_i$. It is straightforward to verify that these maps are functorial.

In type A, this action restricts to an $\mathrm{FI}_A\sharp$–module structure on the ring $H^*(\mathscr{M}_A(n), \mathbb{C})$ generated by the elements $e_{(\mathbf{e}_i - \mathbf{e}_j)^\perp}$. For type D, observe that the inclusion of hyperplane complements $\mathscr{M}_{BC}(n) \hookrightarrow \mathscr{M}_D(n)$ induces an inclusion of cohomology groups $H^*(\mathscr{M}_D(n); \mathbb{C}) \hookrightarrow H^*(\mathscr{M}_{BC}(n), \mathbb{C})$.

The subspaces $H^*(\mathscr{M}_D(n); \mathbb{C}) \subseteq H^*(\mathscr{M}_{BC}(n), \mathbb{C})$ form the B_n–invariant subring generated by the elements $e_{(\mathbf{e}_i - \mathbf{e}_j)^\perp}$ and $e_{(\mathbf{e}_i + \mathbf{e}_j)^\perp}$, $i \neq j$. These inclusions realize $H^*(\mathscr{M}_D(n); \mathbb{C})$ as a sub–$\mathrm{FI}_{BC}\sharp$–module of $H^*(\mathscr{M}_{BC}(n); \mathbb{C})$ generated as an FI_{BC}–algebra by the FI_{BC}–module $H^1(\mathscr{M}_D(n); \mathbb{C})$. Since $H^1(\mathscr{M}_D(n); \mathbb{C})$ is finitely generated in degree ≤ 2, it follows again that $H^*(\mathscr{M}_D(n); \mathbb{C})$ is an $\mathrm{FI}_{BC}\sharp$–algebra of slope 2 with $H^m(\mathscr{M}_D(n); \mathbb{C})$ finitely generated in degree $\leq 2m$. □

Theorem 5.8 has the following consequences.

Corollary 5.9. *In each degree m, the sequence of cohomology groups $\{H^m(\mathscr{M}_{\mathcal{W}}(n), \mathbb{C})\}_n$ of the associated hyperplane complement is uniformly representation stable in degree $\leq 4m$.*

In types A and B/C, Corollary 5.9 recovers [CF13, Theorem 4.1 and 4.6].

Corollary 5.10. *In each degree m, the sequence of characters of the $\mathcal{W}_n$–representations $H^m(\mathscr{M}_{\mathcal{W}}(n), \mathbb{C})$ are given by a unique character polynomial of degree $\leq 2m$ for all n.*

Proof of Corollary 5.10. The statement follows for S_n from [CEF12, Theorem 2.67], and in type B_n from Proposition 4.16. Since the D_n characters are the restriction of the characters of B_n on the B_n–subrepresentations $H^*(\mathscr{M}_D(n); \mathbb{C}) \subseteq H^*(\mathscr{M}_{BC}(n), \mathbb{C})$, these D_n characters are given by the character polynomial for B_n on this sub–$\mathrm{FI}_{BC}\sharp$–module of $H^*(\mathscr{M}_{BC}(\bullet), \mathbb{C})$. □

The character polynomials for $H^m(\mathscr{M}_A(\bullet), \mathbb{C})$ are computed in [CEF12] for some low values of m. The decompositions for $H^1(\mathscr{M}_D(\bullet), \mathbb{C})$ and $H^1(\mathscr{M}_{BC}(\bullet), \mathbb{C})$ are:

$$H^1(\mathscr{M}_D(\bullet), \mathbb{C}) = 2M_D\big(\{\, \ydiagram{2}\,, \varnothing \,\}\big) \qquad \chi_{H^1(\mathscr{M}_D(\bullet), \mathbb{C})} = 2\binom{X_1}{2} + 2\binom{Y_1}{2} + 2X_2$$

$$H^1(\mathscr{M}_{BC}(\bullet), \mathbb{C}) = M_{BC}\big(\, \ydiagram{1}\,, \varnothing \,\big) \oplus M_{BC}\big(\, \ydiagram{2}\,, \varnothing \,\big) \oplus M_{BC}\big(\, \varnothing\,,\, \ydiagram{2} \,\big)$$
$$\chi_{H^1(\mathscr{M}_{BC}(\bullet), \mathbb{C})} = 2\binom{X_1}{2} + 2\binom{Y_1}{2} + 2X_2 + X_1 - Y_1$$

The decompositions for $H^2(\mathscr{M}_D(\bullet), \mathbb{C})$ and $H^2(\mathscr{M}_{BC}(\bullet), \mathbb{C})$ are:

$$H^2(\mathscr{M}_D(\bullet), \mathbb{C}) = M_D\big(\{\, \ydiagram{2}\,, \varnothing \,\}\big) \oplus M_D\big(\{\, \ydiagram{2,1}\,, \varnothing \,\}\big) \oplus M_D\big(\{\, \ydiagram{1}\,, \ydiagram{2} \,\}\big) \oplus M_D\big(\{\, \ydiagram{1}\,, \ydiagram{1,1} \,\}\big)$$
$$\oplus\, M_D\big(\{\, \ydiagram{3,1}\,, \varnothing \,\}\big)^{\oplus 2} \oplus M_D\big(\, \ydiagram{2}\,, + \,\big) \oplus M_D\big(\, \ydiagram{2}\,, - \,\big)$$

$$\chi_{H^2(\mathscr{M}_D(\bullet),\mathbb{C})} = \binom{X_1}{2} - X_1X_2 + \binom{Y_1}{2} + X_2 - Y_2 + 8\binom{X_1}{3} + 8\binom{Y_1}{3} - X_3 - Y_3 + 12\binom{X_1}{4}$$
$$+ 4\binom{X_1}{2}\binom{Y_1}{2} + 12\binom{Y_1}{4} + 4X_2\binom{X_1}{2} + 4X_2\binom{Y_1}{2} - 4\binom{Y_2}{2} - 2Y_4$$

$$H^2(\mathscr{M}_{BC}(\bullet),\mathbb{C}) = M_{BC}(\,\ydiagram{2}\,,\varnothing\,) \oplus M_{BC}(\,\varnothing\,,\ydiagram{2}\,)^{\oplus 2} \oplus M_{BC}(\,\ydiagram{2,1}\,,\varnothing\,)^{\oplus 2}$$
$$\oplus M_{BC}(\,\ydiagram{1}\,,\ydiagram{2}\,)^{\oplus 2} \oplus M_{BC}(\,\ydiagram{1}\,,\ydiagram{1,1}\,) \oplus M_{BC}(\,\ydiagram{3}\,,\varnothing\,) \oplus M_{BC}(\,\varnothing\,,\,\ydiagram{3,1}\,)$$
$$\oplus M_{BC}(\,\ydiagram{3,1}\,,\varnothing\,) \oplus M_{BC}(\,\ydiagram{2}\,,\ydiagram{2}\,)$$

$$\chi_{H^2(\mathscr{M}_{BC}(\bullet),\mathbb{C})} = 3\binom{X_1}{2} + 3\binom{Y_1}{2} - X_1Y_1 + 3X_2 - Y_2 + 14\binom{X_1}{3} + 2\binom{X_1}{2}Y_1$$
$$+ 2X_1\binom{Y_1}{2} + 14\binom{Y_1}{3} + 2X_2X_1 + 2X_2Y_1 - X_3 - Y_3 + 12\binom{X_1}{4} + 4\binom{X_1}{2}\binom{Y_1}{2}$$
$$+ 12\binom{Y_1}{4} + 4X_2\binom{X_1}{2} + 4X_2\binom{Y_1}{2} - 4\binom{Y_2}{2} - 2Y_4$$

Problem 5.11. For each m, compute the character polynomial of the $\mathrm{FI}_A\sharp$–module $H^m(\mathscr{M}_A(\bullet),\mathbb{C})$, and compute its decomposition into induced representations $M_A(U)$. Compute the character polynomials of the $\mathrm{FI}_{BC}\sharp$–modules $H^m(\mathscr{M}_{BC}(\bullet),\mathbb{C})$ and $H^m(\mathscr{M}_D(\bullet),\mathbb{C})$ for each m, and compute the decomposition into induced representations $M_{BC}(U)$.

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